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# A semi-Markov model with geometric renewal processes

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## Abstract

We consider a system modeled by a semi-Markov process where we include geometric renewal process for sojourn times. Pérez-Ocón and Torres-Castro first study this system [1]. In our work here we consider an extended state space for up and down times separately. This allows us to use the standard theory for semi-Markov processes in order to obtain all reliability related measurements as reliability, availability (point and steady-state), mean times and rate of mortality of the system with general initial law. We proceed with a convolution algebra, which allows us to obtain final closed form formulas for the above measurements. Moreover, we present numerical examples.

**Keywords:** semi-Markov process, geometric process, reliability

# 1 Introduction

Semi-Markov processes are important tools in modeling real systems in a very large number of domains [2–5]. Indeed, these processes relax the condition of identically exponentially distributed holding times of continuous time Markov processes which are widely used in reliability engineering [2]. The relaxation of this property permits to model various system behaviors and large range of maintenance strategies.

In this paper, the focus is on the use of a semi-Markov process to deal with reliability measure calculations of a geometric renewal process [6]. The starting point of the work is the paper Pérez-Ocón and Torres-Castro [1] where these authors model a repairable system with internal and external failures. The main problem is that, during the successive failures of this system, the performance level of the system decreases [7]. In fact, after repair, it is not good as new and it is even worse than before. In this case, the above authors make use of the geometric renewal process [8].

As in [1] it is supposed that there are two causes of system failure: external and internal. An external failure occurs following a Poisson process and can be repaired with a fixed probability  $p$ , otherwise the system is replaced by a new one. Therefore, the corrective maintenance in this case is As Good As New (perfect maintenance) with probability  $1 - p$ . After  $N$  repairs, the system is replaced upon failure without repair. Hence the replacement occurs after  $N$  repair or at external failure with probability  $1 - p$ . While the system is operating, it is considered as in up state, while it is in under repair, it's considered in down state. The duration of replacement is considered negligible. After each repair, the system degrades and the repairman is more experienced, modeled by a geometric process [8].

In this paper, each one of the states considered in [1] is split into two states, for example, state  $i$  is split into  $i'$  and  $i''$ , the first one  $i'$  includes the up time (working time) of the system after  $i$  repairs, and state  $i''$  includes the repair of the system after  $i'$ . This scheme, gives us the possibility to calculate all reliability measures known as dependability (reliability, instantaneous and steady-state availability, Mean Time to Failure(MTTF), Mean Time to Repair(MTTR), Mean Up Time(MUT), Mean Down Time(MDT), rate of occurrence of failures(ROCOF), etc.)in Markov scheme as presented in [9, 10], in the standard semi-Markov scheme, as presented in [2, 11, 12]. Moreover we obtain closed form solution formulas for all measures.

The paper is organized as follows. Section 2 presents the model, which includes assumptions, semi-Markov kernel, and the transition kernel of the Embedded Markov Chain (EMC). Section 3 presents reliability measures, including reliability, availability, Mean Times and ROCOF. Section 4 presents two numerical examples with formulas presented in this article.

## 2 The Model

Let's consider a system with only one component having two states up and down. The component is in the up state while operating, and in the down state while is being repaired. The system satisfies the following assumptions.

### 2.1 Introduction of the model

Let us consider a semi-Markov process (SMP)  $(Z_t)_{t \geq 0}$ , with finite state space  $E$ , and the embedded Markov renewal process (MRP)  $(J_n, S_n)_{n \in \mathbb{N}}$ , where  $(J_n)_{n \in \mathbb{N}}$  is the Embedded Markov Chain (EMC) and  $S_0 = 0 < S_1 < S_2 < \dots < S_n < \dots$  the jump times where  $(Z_t)_{t \geq 0}$  changes the states. We have:

$$Z_t = J_n, \quad S_n \leq t < S_{n+1}$$

Define also the semi-Markov kernel

$$Q_{ij}(x) = \mathbb{P}(J_{n+1} = j, S_{n+1} - S_n \leq x | J_n = i)$$

for  $i, j \in E, x \geq 0$ . We consider here the standard SMP with  $Q_{ii}(x) \equiv 0$ . Pour plus de détails sur les semi-Markov processes see references [2–5]. For more definitions and results on semi-Markov processes, see references [9, 10] concern the Markov case.

### 2.2 Assumptions of the model

The following assumptions are formulated as in [1], except that in this paper each state is split into two states: up state (operational) and down state (under repair). Instead of  $N + 1$  states in [1], we have  $2N + 1$  states. The difference is situated in Assumption 7.

#### *Assumption 1*

There are two system failure causes: internal and external. The external failure occurs following a Poisson process with rate  $\lambda$ . Bernoulli trials determine whether these failures are repairable with probability  $p$ , or non-repairable with a probability  $q$  ( $q = 1 - p$ ). Internal failure occurs by aging, which is always non-repairable.

#### *Assumption 2*

All failures occur independently of each other.

#### *Assumption 3*

The system is replaced by a new one when a non-repairable failure occurs or after  $N$  repairs. We consider the replacement as a perfect maintenance which restores the system to its initial state. Moreover, the replacement is instantaneous.

*Assumption 4*

Let set  $U_n$  as the lifetime of the system after its  $n$  repairs. The lifetime of a new system has Cumulative distribution Function  $F$ . The system deteriorates after each repair and the distribution function of  $U_n$  is given by

$$F_n(x) = \mathbb{P}(U_n \leq x) = F(a^n x), x \geq 0$$

where  $a$  is called *operational factor*,  $a > 0$ .  $F$  has a finite mean.

*Assumption 5*

A repairman deals with repairable failures. Let set  $D_n$  the repair time after its  $n^{th}$  failure. Let  $G$  be the Cumulative Distribution Function of the repair time of the first failure. The distribution function of  $n^{th}$  repair is given by

$$G_n(x) = \mathbb{P}(D_n \leq x) = G(b^n x), x \geq 0$$

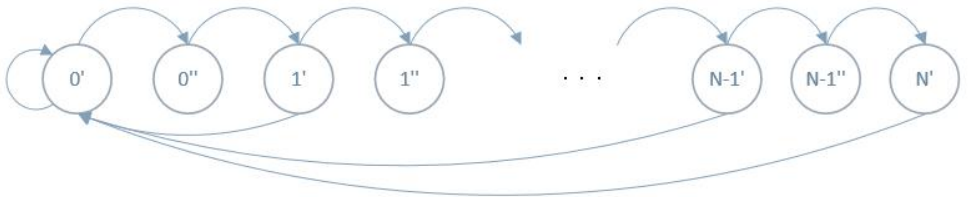
where  $b$  is called *repair factor*,  $b > 0$ ;  $G$  has finite mean.

*Assumption 6*

Sequences  $\{U_n, n \geq 1\}$  and  $\{D_n, n \geq 1\}$  are independent, and each one of both sequences is independent but not identically distributed.

*Assumption 7*

The states of the system are  $E = \{0', 0'', 1', 1'', \dots, N-1', N-1'', N'\}$ , which contains up states  $U = \{0', 1', 2', \dots, N'\}$  and down states  $D = \{0'', 1'', 2'', \dots, N-1''\}$ , where the state  $0'$  refers to the perfect state when the system is new or replaced as good as new. This system can be represented by the following graph in Figure 1.



**Fig. 1** State transition graph of the system

### 2.3 The Semi-Markov Kernel

Based on the previous assumptions and the graph in Figure 1, we introduce the Semi-Markov kernel of this process, presented by matrix function  $Q(x), x \geq 0$ .

$$Q(x) = \left( \begin{array}{cccc|cccc}
 0 & 0 & 0 & \dots & 0 & \frac{Q_{0'0''}(x)}{1-P_{0'0'}} & 0 & 0 & \dots & 0 \\
 Q_{1'0'}(x) & 0 & 0 & \dots & 0 & 0 & Q_{1'1''}(x) & 0 & \dots & 0 \\
 Q_{2'0'}(x) & 0 & 0 & \dots & 0 & 0 & 0 & Q_{2'2''}(x) & \dots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 Q_{N-1'0'}(x) & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & Q_{N-1'N-1''}(x) \\
 Q_{N'0'}(x) & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\
 \hline
 0 & Q_{0''1'}(x) & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
 0 & 0 & Q_{1''2'}(x) & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & 0 & \dots & Q_{N-1''N'}(x) & 0 & 0 & 0 & \dots & 0
 \end{array} \right)$$

$$= \left( \begin{array}{c|c}
 Q_0 & Q_{01} \\
 \hline
 Q_{10} & Q_1
 \end{array} \right)$$

Where  $Q_0$  includes transitions from up states U to up states U;  $Q_1$  includes transitions from down states D to down states D;  $Q_{01}$  includes transitions from up states U to down states D;  $Q_{10}$  includes transitions from down states D to up states U.

**Remark**

Here a standard SM kernel is built, which means that  $Q_{ii}(t) = 0$ . The difference with the initial SM kernel, say  $\check{Q}$ , is only in the first line and in particular in the entries  $(0', 0')$  and  $(0', 0'')$ . For the initial SM kernel,  $\check{Q}_{0'0'}(x) \neq 0$  and  $\check{Q}_{0'0''}(x) \neq 0$ , while for the standard form:  $Q_{0'0'}(x) \equiv 0$  and  $Q_{0'0''}(x) = \frac{\check{Q}_{0'0''}(x)}{1-P_{0'0'}}$ . The other terms are the same.

The only way to leave state  $i'$  to state  $i''$ , is that the system fails externally, and the failure can be repaired, which means

$$Q_{i'i''}(x) = \int_0^x \lambda e^{-\lambda t} p(1 - F(a^{i'}t)) dt, \quad i' \in \{0, \dots, (N-1)'\}$$

The system is replaced by a new one after an internal failure or an non-repairable external failure, it comes to state  $0'$  with the following probability

$$Q_{i'0'}(x) = \int_0^x \lambda e^{-\lambda t} (1-p)(1-F(a^{i'}t)) dt + \int_0^x e^{-\lambda t} dF(a^{i'}t), \quad i' \in \{0, \dots, (N-1)'\}$$

To get to the state  $i''$  from  $i+1'$ , a repairman interferes, which leads to

$$Q_{i''i+1'}(x) = G(b^{i'}x), \quad i' \in \{0, \dots, (N-1)'\}$$

After  $N$  repairs the system is in state  $N'$ , the system will be replaced directly upon failure with the following probability:

$$Q_{N'0'}(x) = \int_0^x \lambda e^{-\lambda t} (1 - F(a^{N'}t)) dt + \int_0^x e^{-\lambda t} dF(a^{N'}t)$$

### 2.4 Embedded Markov Chain

The EMC  $\{J_n, n \geq 0\}$  gives the successive visited states by the SMP  $(Z_t)_{t \geq 0}$  after  $n^{th}$  jump. Its transition matrix  $\mathbf{P} = (P_{ij}, i, j \in E)$  can be obtained by taking limits:  $P_{ij} = \lim_{x \rightarrow \infty} Q_{ij}(x) = Q_{ij}(\infty)$ .

$$\mathbf{P} = \left( \begin{array}{cccc|cccc} 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ P_{1'0'} & 0 & 0 & \dots & 0 & 0 & P_{1'1''} & 0 & \dots & 0 \\ P_{2'0'} & 0 & 0 & \dots & 0 & 0 & 0 & P_{2'2''} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ P_{N-1'0'} & 0 & 0 & \dots & 0 & 0 & \dots & 0 & P_{N-1'N-1''} & \\ \hline 1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 0 \end{array} \right)$$

$$= \left( \begin{array}{c|c} P_0 & P_{01} \\ \hline P_{10} & P_1 \end{array} \right)$$

In this section, the stationary distribution of  $\{J_n, n \geq 0\}$  which is not developed in [1]. The model proposed by Pérez-Ocón and Torres-Castro [1], by merging states  $(i', i'')$  to  $i$  and denoting  $N'$  by  $N$ . In this case, the stationary distribution can be defined for the EMC on the state space  $S = \{0, 1, \dots, N\}$  with the transition matrix is then

$$\left( \begin{array}{cccc|c} P_{0,0} & P_{0,1} & 0 & \dots & 0 \\ P_{1,0} & 0 & P_{1,2} & & 0 \\ P_{2,0} & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ P_{N-1,0} & 0 & 0 & \dots & P_{N-1,N} \\ P_{N,0} & 0 & 0 & \dots & 0 \end{array} \right)$$

Its stationary distribution is then given by a row vector  $\mathbf{v} = (v_0, v_1, \dots, v_N)$ , obtained by solving the usual equation  $\mathbf{v}P = \mathbf{v}$  with  $\sum_{i \in E} v_i = 1$ . Hence,

$$\begin{cases} v_0 = \frac{1}{1 + \sum_{j=1}^N \prod_{k=1}^j P_{k-1,k}} \\ v_i = \frac{\prod_{j=1}^i P_{j-1,j}}{1 + \sum_{j=1}^N \prod_{k=1}^j P_{k-1,k}}, \quad i = 1, \dots, N \end{cases} \quad (1)$$

Let be  $\boldsymbol{\rho} = (\rho_{0'}, \dots, \rho_{N'}, \rho_{0''}, \dots, \rho_{N-1''})$  the the stationary distribution of  $\{J_n, n \geq 0\}$ . Since,

$$\rho_{i'} + \rho_{i''} = v_i, \quad i = 0, 1, \dots, N \quad (2)$$

then  $\boldsymbol{\rho} = (\rho', \rho'')$ , is obtained by solving Eq(2) as follows:

$$\begin{cases} \rho_{i'} = \frac{v_i}{1 + P_{i'i''}} & i = 1, \dots, N - 1 \\ \rho_{i''} = P_{i'i''} \rho_{i'} = \frac{P_{i'i''} v_i}{1 + P_{i'i''}} & i = 1, \dots, N - 1 \\ \rho_{N'} = v_N & i = N \end{cases} \quad (3)$$

### 3 Reliability

The initial law of the process  $(Z_t)_{t \geq 0}$  is denoted by the row vector

$$\boldsymbol{\alpha} = (\alpha_{0'}, \dots, \alpha_{N'}, \alpha_{0''}, \dots, \alpha_{N-1''}) = (\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1)$$

Let us define the distribution function of sojourn time in state  $i$ ,  $H_i(t) = \sum_{j \in E} Q_{ij}(t) = 1 - \overline{H}_i(t)$ ,  $i \in U$ . In our case, we write the vector function  $H(t)$  as follows

$$\overline{H}(t) = \begin{pmatrix} \overline{H}_0(t) \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{H}_{0'}(t) \\ \vdots \\ \overline{H}_{N'}(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - \frac{Q_{0'0''}(t)}{1 - P_{0'0''}} \\ 1 - Q_{0'1'}(t) - Q_{1'1''}(t) \\ \vdots \\ 1 - Q_{N-2'N-1'}(t) - Q_{N-1'N-1''}(t) \\ 1 - Q_{N'0'}(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Define also the Stieltjes convolution of a function  $\varphi : E \times R_+ \rightarrow R_+$  by  $Q$  as follows

$$Q * \varphi(i, t) = \sum_{j \in E} \int_0^t Q_{ij}(ds) \varphi(j, t - s)$$



From this definition, the  $n$ -fold convolution of  $Q$  by itself can be derived by the following expression:

$$Q_{ij}^{(n)}(t) = \sum_{k \in E} \int_0^t Q_{ik}(ds) Q_{kj}^{(n-1)}(t-s), \quad n \geq 1$$

and

$$\begin{cases} Q_{ij}^{(0)}(t) = \mathbb{1}_{R_+}(t) \delta_{ij} \\ Q_{ij}^{(1)}(t) = Q_{ij}(t) \end{cases}$$

where  $\delta_{ij}$  is the Kronecker's symbol which means  $\delta_{ij} = 1$ , if  $i = j$  and 0 otherwise. As in [2], the Stieltjes convolution algebra is used with neutral element

$$I := I(t) = \begin{cases} I & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

where  $I$  is the identity matrix.

### 3.1 The Reliability function

From [2], we know that the reliability function of the system is:

$$R(t) = \alpha_0 (I - Q_0)^{(-1)} * \overline{H_0}(t)$$

where  $(I - Q_0)^{(-1)}$  means the inverse of  $I - Q_0$  in the convolution sense. After algebraic calculation in the convolution, we obtain the reliability function

$$R(t) = \sum_{i=0}^N \alpha_{i'} \overline{H_{0'}}(t) + \sum_{i=1}^N \alpha_{i'} Q_{i'0'} * \overline{H_{i'}}(t)$$

If the system begins surely at state  $0'$ , which means  $\alpha_0 = (1, 0, \dots, 0)$ , the reliability of the system is simply

$$R(t) = \overline{H_{0'}}(t) = 1 - \frac{Q_{0'0'}(t)}{1 - P_{0'0'}}$$

### 3.2 Availability

Here we present first the stationary distribution  $\pi$  of the SMP  $(Z_t)_{t \geq 0}$  as follows, see, e.g, [2]

$$\pi_i = \frac{\rho_i m_i}{m}$$

where

$$m = \sum_{i \in E} \rho_i m_i$$

We can now calculate the instantaneous availability by [2]

$$A(t) = \alpha (I - Q)^{(-1)} * \overline{H}(t)$$

To calculate  $(I - Q)^{(-1)}$ , we here use the inversion of matrix in the convolution sens by block formula. Put  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ , where A and D are square bloc matrices function of  $t$ .

As in our case, we have  $D = I$ , then

$$\begin{aligned} M^{(-1)} &= \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{U} \end{pmatrix} \\ &= \begin{pmatrix} X^{(-1)} & -X^{(-1)} * B \\ -C * X^{(-1)} & I + C * X^{(-1)} * B \end{pmatrix} \end{aligned}$$

Where  $X := A - B * C = I - Q_0 - Q_{01} * Q_{10}$

Based on the previous representation we get the inverse of  $(I - Q)$  as follows

$$\det(X) = 1 - \sum_{k=1}^N \frac{Q_{k'0'}}{1 - P_{0'0'}} * \prod_{l=0}^{k-1} Q_{l'l''} * Q_{l''l+1'}$$

The transpose of the convolution of  $\mathbf{X}$  is

$$com^{\top} X = \begin{pmatrix} 1 & X_{01} & \cdots & X_{0j} & \cdots & X_{0N} \\ X_{10} & 1 - X_{11} & \cdots & 1 - X_{1j} & \cdots & 1 - X_{1N} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ X_{i0} & X_{i1} & \cdots & 1 - X_{ii} & \cdots & 1 - X_{iN} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ X_{N0} & X_{N1} & \cdots & X_{Nj} & \cdots & 1 - X_{NN} \end{pmatrix}$$

Where

$$X_{ij} = \begin{cases} \sum_{k=i+1}^{j-1} \frac{Q_{k'0'}}{1 - P_{0'0'}} * \prod_{l \in \llbracket 0, k-1 \rrbracket \setminus \{i\}} Q_{l'l''} * Q_{l''l+1'} & , j > i > 1 \\ \sum_{k \in \llbracket 1, N \rrbracket \setminus \{i\}} \frac{Q_{k'0'}}{1 - P_{0'0'}} * \prod_{l \in \llbracket 0, k-1 \rrbracket \setminus \{i\}} Q_{l'l''} * Q_{l''l+1'} & , i = j > 1 \\ \sum_{k \in \llbracket j+1, N \rrbracket} \frac{Q_{k'0'}}{1 - P_{0'0'}} * \prod_{l \in \llbracket 0, k-1 \rrbracket \setminus \{i\}} Q_{l'l''} * Q_{l''l+1'} & , i > j > 1 \\ \frac{\prod_{l=0}^{j-1} Q_{l'l''} * Q_{l''l+1'}}{1 - P_{0'0'}} & , i = 0, j > 0 \\ \sum_{k=1}^N Q_{k'0'} * \prod_{l=j}^{k-1} Q_{l'l''} * Q_{l''l+1'} & , i > 0, j = 0 \\ \sum_{k=1}^{j-1} Q_{k'0'} * \prod_{l=1}^{k-1} Q_{l'l''} * Q_{l''l+1'} & , i = 1, j > 1 \\ \sum_{k=2}^N Q_{k'0'} * \prod_{l=2}^{k-1} Q_{l'l''} * Q_{l''l+1'} & i = j = 1 \\ 0 & i = j = 0 \end{cases}$$

Hence,

$$\mathbf{X} = (\det(X))^{(-1)} * com^{\top} X$$

And

$$-C * com^T X = \begin{pmatrix} Q_0''_{1'} * X_{10} & Q_0''_{1'} * (1 - X_{11}) & \cdots & Q_0''_{1'} * (1 - X_{1j}) & \cdots & Q_0''_{1'} * (1 - X_{1N}) \\ Q_1''_{2'} * X_{20} & Q_1''_{2'} * X_{21} & \cdots & Q_1''_{2'} * (1 - X_{2j}) & \cdots & Q_1''_{2'} * (1 - X_{2N}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ Q_{i''_{i+1}'} * X_{i+1,0} & Q_{i''_{i+1}'} * X_{i+1,1} & \cdots & Q_{i''_{i+1}'} * (1 - X_{i+1,j}) & \cdots & Q_{i''_{i+1}'} * (1 - X_{i+1,N}) \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ Q_{N-1''_{N'}} * X_{N0} & Q_{N-1''_{N'}} * X_{N1} & \cdots & Q_{N-1''_{N'}} * X_{Nj} & \cdots & Q_{N-1''_{N'}} * (1 - X_{NN}) \end{pmatrix}$$

Hence,

$$\mathbf{Z} = (\det(X))^{(-1)} * (-C * com^T X)$$

We calculate  $-com^T X * B$  as follows

$$-com^T X * B = \begin{pmatrix} \frac{Q_0'_{0''}}{1-P_0'_{0''}} & Q_{1'1''} * X_{01} & \cdots & Q_{j'j''} * X_{0j} & \cdots & Q_{N-1'N-1''} * X_{0,N-1} \\ \frac{Q_0'_{0''}}{1-P_0'_{0''}} * X_{10} & Q_{1'1''} * (1 - X_{11}) & \cdots & Q_{j'j''} * (1 - X_{1j}) & \cdots & Q_{N-1'N-1''} * (1 - X_{1,N-1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{Q_0'_{0''}}{1-P_0'_{0''}} * X_{i0} & Q_{1'1''} * X_{i1} & \cdots & Q_{j'j''} * (1 - X_{ij}) & \cdots & Q_{N-1'N-1''} * (1 - X_{i,N-1}) \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ \frac{Q_0'_{0''}}{1-P_0'_{0''}} * X_{N-1,0} & Q_{1'1''} * X_{N-1,1} & \cdots & Q_{j'j''} * X_{N-1,j} & \cdots & Q_{N-1'N-1''} * (1 - X_{N-1,N-1}) \\ \frac{Q_0'_{0''}}{1-P_0'_{0''}} * X_{N0} & Q_{1'1''} * X_{N1} & \cdots & Q_{j'j''} * X_{Nj} & \cdots & Q_{N-1'N-1''} * X_{N,N-1} \end{pmatrix}$$

Hence,

$$\mathbf{Y} = \det(X)^{(-1)} * (-com^T X) * B$$

$$\mathbf{U} = I + C * X * B =$$

$$\begin{pmatrix} 1 + Q_0''_{1'} * \frac{Q_0'_{0''}}{1-P_0'_{0''}} * X_{10} & \cdots & Q_0''_{1'} * Q_{j'j''} * (1 - X_{1j}) & \cdots & Q_0''_{1'} * Q_{N-1'N-1''} * (1 - X_{1,N-1}) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ Q_{i-1''_{i'}} * \frac{Q_0'_{0''}}{1-P_0'_{0''}} * X_{i-1,0} & \cdots & Q_{i-1''_{i'}} * Q_{j'j''} * X_{i-1,j} & \cdots & Q_{i-1''_{i'}} * Q_{N-1'N-1''} * (1 - X_{i-1,N-1}) \\ \vdots & & \vdots & \ddots & \vdots \\ Q_{N-1''_{N'}} * \frac{Q_0'_{0''}}{1-P_0'_{0''}} * X_{N-1,0} & \cdots & Q_{N-1''_{N'}} * Q_{j'j''} * X_{N-1,j} & \cdots & Q_{N-1''_{N'}} * Q_{N-1'N-1''} * X_{N-1,N-1} \end{pmatrix}$$

After all the above calculations, we finally get the availability formula:

$$\begin{aligned}
 A(t) = & \alpha_{0'} (\det(X))^{(-1)} * (\overline{H_{0'}} + \sum_{j=1}^N X_{0j} * \overline{H_{j'}})(t) \\
 & + \sum_{i=1}^N \alpha_{i'} (\det(X))^{(-1)} * (\sum_{j=0}^{i-1} X_{ij} * \overline{H_{j'}} + \sum_{i=1}^N (1 - X_{ij}) * \overline{H_{j'}})(t) \\
 & + \alpha_{0''} (\det(X))^{(-1)} * (\frac{Q_{0'0''}}{1 - P_{0'0''}} * \overline{H_{0''}} + \sum_{j=1}^{N-1} Q_{j'j''} * X_{0j} * \overline{H_{j''}})(t) \\
 & + \sum_{i=1}^{N-1} \alpha_{i''} (\det(X))^{(-1)} * \frac{Q_{0'0''}}{1 - P_{0'0''}} * X_{i0} * \overline{H_{0''}}(t) \\
 & + \sum_{i=1}^{N-1} \alpha_{i''} (\det(X))^{(-1)} * Q_{j'j''} * (\sum_{j=0}^{i-1} X_{ij} * \overline{H_{j''}} + \sum_{i=1}^N (1 - X_{ij}) * \overline{H_{j''}})(t)
 \end{aligned}$$

In particular, if the system begins with the perfect state, the previous formula gives

$$A(t) = (\det(X))^{(-1)} * (\overline{H_{0'}} + \sum_{j=1}^N X_{0j} * \overline{H_{j'}})(t)$$

We can also get the steady-state availability as limit of the pointwise availability  $A(t)$  as  $t$  goes to infinity [2]

$$\bar{A} = \sum_{i \in U} \pi_i = \frac{\sum_{i \in U} \rho_i m_i}{m}$$

### 3.3 Mean times

Let define the mean sojourn time vector  $\mathbf{m}$ :

$$\mathbf{m} = \begin{pmatrix} \mathbf{m}_0 \\ \mathbf{m}_1 \end{pmatrix} = \begin{pmatrix} m_{0'} \\ \vdots \\ m_{N'} \\ m_{0''} \\ \vdots \\ m_{N-1''} \end{pmatrix}$$

Here we introduce  $\mathbf{m}_0$  explicitly

$$\mathbf{m}_0 = \begin{pmatrix} \int_0^\infty [1 - \frac{Q_{0'0''}}{1-P_{0'0'}}(t)]dt \\ \int_0^\infty [1 - Q_{1'0'}(t) - Q_{1'1''}(t)]dt \\ \vdots \\ \int_0^\infty [1 - Q_{N-1'0'}(t) - Q_{N-1'N-1''}(t)]dt \\ \int_0^\infty [1 - Q_{N'0'}(t)]dt \end{pmatrix}$$

The formulas of several mean times are presented in [12]:

$$\begin{aligned} MTTF &= \alpha_0(I - P_0)^{-1}\mathbf{m}_0 \\ MTTR &= \alpha_1(1 - P_1)^{-1}\mathbf{m}_1 \\ MUT &= \frac{\pi_0\mathbf{m}_0}{\pi_1P_{10}\mathbb{1}_r} \\ MDT &= \frac{\pi_1\mathbf{m}_1}{\pi_0P_{01}\mathbb{1}_{d-r}} \end{aligned}$$

After calculation, we obtain:

$$\begin{aligned} MTTF &= \alpha_{0'}m_{0'} + \sum_{i=1}^N \alpha_{i'0'}(m_{0'}P_{i'0'} + m_{i'}) \\ MTTR &= \sum_{i=0}^{N-1} \alpha_{i''}m_{i''} \\ MUT &= \frac{\sum_{i=0}^N \rho_{i'}m_{i'}^2}{\sum_{i=0}^{N-1} \rho_{i''}m_{i''}} \\ MDT &= \frac{\sum_{i=0}^{N-1} \rho_{i''}m_{i''}^2}{\sum_{i=0}^N \rho_{i'}m_{i'}} \end{aligned}$$

If the system begins surely at state  $0'$ , which means  $\alpha_0 = (1, 0, \dots, 0)$ , we have the mean time to failure

$$MTTF = m_{0'}$$

And for  $\alpha_1 = (1, 0, \dots, 0)$ , the mean time to repair is

$$MTTR = m_{0''}$$

### 3.4 Rate of Occurrence of Failure

Devote by  $N_f(t)$  the counting process of transitions from Up states  $U$  to down states  $D$  at time  $t$ , so  $\mathbb{E}[N_f(t)]$  is the expected number of failures that have occurred by time  $t$ . Then the ROCOF,  $ro(t)$ , at time  $t$ , is defined by

$$ro(t) := \frac{d}{dt} \mathbb{E}[N_f(t)]$$

For semi-Markov process the ROCOF is obtained in [13], for several semi-Markov processes is [11], for Markov processes is [10].

Here we will suppose that the semi-Markov kernel  $(Q_{ij})$  has derivatives (Radon-Nikodym)

$$q_{ij}(t) = \frac{d}{dt} Q_{ij}(t)$$

According to [13], the ROCOF of a SM system is given by the formula

$$ro(t) = \sum_{i \in U} \sum_{j \in D} \sum_{l \in E} \int_0^t \alpha_l \psi_{li}(du) q_{ij}(t-u) \tag{4}$$

The Markov renewal function is expressed as

$$\psi(t) := (I - Q)^{(-1)}(t)$$

If we put the Markov renewal function  $\psi$  in bloc matrix form, following the partition U and D of E, we have

$$\psi(t) = \begin{pmatrix} \psi_0 & \psi_{01} \\ \psi_{10} & \psi_1 \end{pmatrix} (t)$$

The representation of ROCOF from (3) can be written as

$$ro(t) = \alpha_0 \psi_0 * q_{01}(t) \mathbb{1}_{d-r} + \alpha_1 \psi_{10} * q_{01}(t) \mathbb{1}_{d-r} \tag{5}$$

where  $\mathbb{1}_{d-r} = (1, \dots, 1)^\top$  with  $d - r$  ones.

If the system begins in its perfect state as explained before, then we get

$$ro(t) = (\det(X))^{(-1)} * \left( \frac{q_{0'0''}}{1 - P_{0'0''}} + \sum_{j=1}^{N-1} X_{0j} * q_{j'j''} \right)$$

In our case, the SMP is irreducible and we suppose that the mean sojourn times are finite, then we have [13]

$$\lim_{t \rightarrow \infty} ro(t) = \sum_{i \in U} \sum_{j \in D} \frac{P_{ij}}{\mu_{ii}}$$

where  $\mu_{ii}$  is the mean recurrence time to state  $i \in E$ , for the SMP  $(Z_t)$

$$\mu_{ii} = \frac{m}{\rho_i}$$

So we obtain the limit of ROCOF

$$\lim_{t \rightarrow \infty} ro(t) = \frac{\sum_{i \in U} \rho_i P_{i''}}{m}$$

## 4 Numerical examples

### 4.1 Example 1

To compare with the results in [1], we have chosen the same parameters along with same distributions assumed as follows. The external failure occurs following a Poisson process with rate  $\lambda = 0.02/h$ . External failures can be repairable with a probability  $p = 0.87$ , or be non-repairable with a probability  $q = 1 - p = 0.13$ . The lifetime of the system follows Phase-type distributions (PH distribution [14]) with representation  $(\alpha, a^i \mathbf{T}), i = 0, 1, \dots, N$ . The repairman deals with repairable external failure with a PH distribution with representation  $(\beta, b^i \mathbf{S})$ . The deterioration after each repair is  $a = 1.25$ , and the coefficient of skill is  $b = 0.9$ . The system begins in perfect state 0', and is replaced directly after 3 repairs (N=3), this implies that the total number of states are 7 with 4 Up states.

Matrix  $\mathbf{T}$  and  $\mathbf{S}$  are given by

$$\mathbf{T} = \begin{pmatrix} -0.001 & 0.001 & 0 \\ 0 & -0.08 & 0.08 \\ 0 & 0 & -0.01 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0.01 & -0.08 & 0.07 \\ 0.005 & 0 & -0.2 \end{pmatrix}$$

With formulas presented in this article, we get the transition matrix

$$P = \left( \begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.1464 & 0 & 0 & 0 & 0 & 0.8536 & 0 \\ 0.1538 & 0 & 0 & 0 & 0 & 0 & 0.8462 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right)$$

Then the stationary law of the EMC is

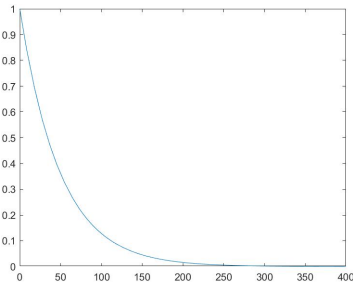
$$\rho = (0.1556 \ 0.1442 \ 0.1236 \ 0.1931 | 0.1556 \ 0.1231 \ 0.1046)$$

The mean sojourn time in each state is

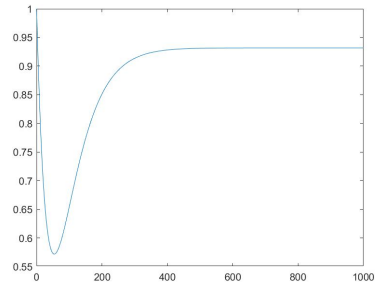
$$\mathbf{m} = \begin{pmatrix} 48.8302 \\ 49.0573 \\ 48.6301 \\ 48.6301 \\ 4.0816 \\ 4.5351 \\ 5.0391 \end{pmatrix}$$

and  $m = 31.6885$ . Finally from the above formulas  $MTTF = 64.5150$ ,  $MTTR = 3.4140$ ,  $MUT = 78.1686$ ,  $MDT = 4.4885$ ,  $\bar{A} = 0.9286$ ,  $\lim_{t \rightarrow \infty} ro(t) = 0.0159$

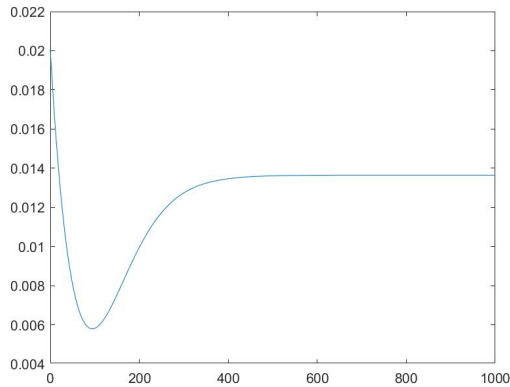
Considering a perfect initial state, in Figures 2, 3 and 4, the reliability, the instantaneous availability and the point-wise rate of occurrence are depicted respectively. If the initial law of system follows uniform distribution, which means  $\alpha_0 = (\frac{1}{N+1} \cdots \frac{1}{N+1})$ , therefore  $MTTF = 36.4347$ ,  $MTTR = 3.4140$  and the reliability function is presented in Figure 5



**Fig. 2** Reliability with initial perfect state

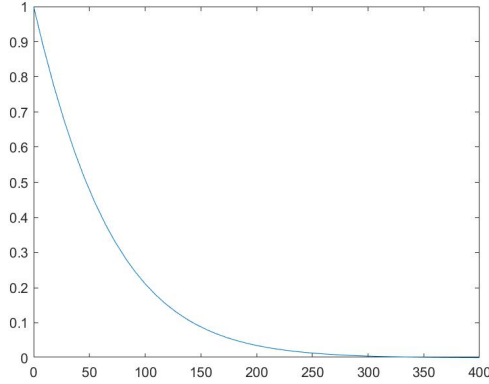


**Fig. 3** Availability with initial perfect state



**Fig. 4** ROCOF with initial perfect state





**Fig. 5** Reliability with Uniform initial law

### 4.2 Example 2

We have implemented the model in Matlab program [14] for numerical examples. The parameters are assumed as follows. The external failure occurs following a Poisson process with rate  $\lambda = 0.02/h$ , which means the arrival meantime is 500h. External failures can be repairable with a probability  $p = 0.87$ , or be non-repairable with a probability  $q = 1 - p = 0.13$ . We suppose that the lifetime of the system follows a Weibull distribution,  $F(x) = 1 - \exp(-(\frac{x}{\alpha})^\beta)$  with parameter  $\alpha = 1, \beta = 1.5$ , which shows that the system degrades by time. The repairman deals with repairable external failure with a Weibull distribution function, where  $\alpha_1 = 1, \beta_1 = 0.9$ , which means the repairman treats failures faster each time. The coefficient of deterioration after each repair is  $a = 1.25$ , and the coefficient of skill is  $b = 0.9$ . We suppose the system begins in perfect state, and is replaced directly after 3 repairs ( $N = 3$ ), this implies that the total number of states are 7 with 4 Up states. With formulas presented in this article, we get the transition matrix of the EMC

$$P = \left( \begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.9876 & 0 & 0 & 0 & 0 & 0.0124 & 0 \\ 0.9900 & 0 & 0 & 0 & 0 & 0 & 0.0100 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right)$$

Then the stationary law of the EMC is

$$\rho = ( 0.4923 \ 0.0151 \ 0.0002 \ 0.0000 | 0.4923 \ 0.0002 \ 0.0000 )$$

The mean sojourn time in each state is

$$\mathbf{m} = \begin{pmatrix} 0.6584 \\ 0.7146 \\ 0.5729 \\ 0.4591 \\ \frac{1.0522}{1.1691} \\ 1.2990 \end{pmatrix}$$

and  $m = 0.8532$ . Moreover, from the above formulas  $MTTF = 0.6584$ ,  $MTTR = 0$ ,  $MUT = 0.6803$ ,  $MDT = 1.0522$ ,  $\bar{A} = 0.3927$ ,  $\lim_{t \rightarrow \infty} ro(t) = 0.5770$ .

In Figure 6, for a perfect initial state, the reliability is depicted respectively. If the initial law of system follows an uniform distribution, which means  $\alpha = (\frac{1}{N+1} \cdots \frac{1}{N+1})$ , different mean time to failure and mean time to repair are obtained  $MTTF = 0.6237$ ,  $MTTR = 0.8801$

The reliability function with an uniform initial distribution is presented in Figure 7

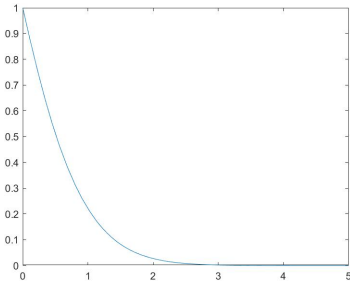


Fig. 6 Reliability with initial perfect state

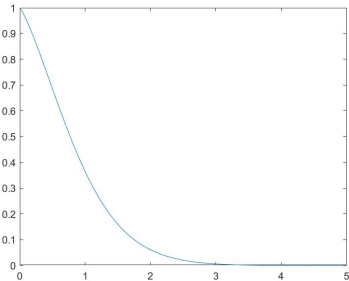


Fig. 7 Reliability with Uniform initial law

## 5 Concluding Remarks

The extension of state space presented in this article allows us to obtain all reliability indicators (transient and steady-state) in a standard way. The way of convolution algebra used here allows us to obtain closed form solution for reliability and steady-state indicators. Another way to obtain numerical results is to consider the Markov renewal function  $\psi(t)$  as a series, i.e.

$$\psi(t) = \sum_{n \geq 0} Q^{(n)}(t)$$

In that case, a truncation test is needed, see [2] and [15]. An additional future that we can face here, is to consider not instantaneous replacement of a new

unit, but with a time duration, deterministic or stochastic. For this we can just consider an additional state for this operation, of course, this consideration leaves unchanged reliability and MTTF, while the other indicators will be affected.

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