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PERFORMANCE OF PROGNOSIS INDICATORS FOR SUPERIMPOSED RENEWAL PROCESSES

XINGHENG LIU 

The Laboratory of Systems Modelling and Dependability, Université de Technologie de Troyes, Troyes, France;

Department of Mechanical and Industrial Engineering, Norwegian University of Science and Technology, Trondheim, Norway

YANN DIJOUX 

The Laboratory of Systems Modelling and Dependability, Université de Technologie de Troyes, Troyes, France.

E-mail: yann.dijoux@utt.fr

JØRN VATN

Department of Mechanical and Industrial Engineering, Norwegian University of Science and Technology, Trondheim, Norway

HÅKON TOFTAKER

Monitoring and Analysis, Bane NOR SF, Hamar, Norway

The paper deals with prognosis estimation for industrial systems in a series configuration, modeled by superimposed renewal processes (SRP), when the cause of failures is not available. In the presence of missing information, an SRP is commonly approximated by a Poisson process or a virtual age model. The performance of the approximations was assessed in the ideal configuration where all parameters of the models are known. The current article adopts a practitioner's perspective by assuming that the parameters of the models are unknown and must be estimated. In addition to inference procedures, the assessment of the prognosis indicators, such as the remaining useful life, is discussed. Finally, we investigate a fleet of infrastructure components of the Norwegian railway network operated by Bane NOR.

Keywords: applied probability, reliability theory, simulation, stochastic modelling

1. INTRODUCTION

In this paper, the superposition of independent renewal processes (SRP) is discussed in the context of reliability engineering [31,35,36] where it can be referred to as partial repair

model [3,2]. A maintainable system under partial repair is a series system such that after a failure, the failed component is automatically identified and instantaneously replaced by a new one of the same technology (as good as new). The non-failed components are assumed to be minimally maintained (as bad as old) which implies that their respective failure rate is the same before and after the maintenance activity. At a system level, this partial repair is also characterized as imperfect repair [8,32]. The components are assumed to be statistically independent and can represent either physical units or virtual ones [24] associated with an independent failure mode.

Maintenance records are often incomplete, and the information on the identity of the failed components is not necessarily available. Consequently, the observations are commonly reduced to a pooled output [7], consisting of the failure times at a system level. Assessing the health of the system such as its overall aging and the maintenance efficiency usually starts off by estimating model parameters. When the number of components is known and all components are identical, inference procedures have been carried out by Zhang *et al.* [36] directly from an SRP using computational partitioning. Nevertheless, when the number of pooled events is limited, or when the components are different from each other, direct inference methods are hard to implement, which inspires diverse approximation approaches.

Alternately, an SRP is approximated by a homogeneous Poisson process [27], a renewal process [34], an imperfect maintenance model [21,35] or a copula model [22]. The benefits of preferring imperfect maintenance models or copulas to a renewal process, when the parameters of the approximated SRP are known, have been highlighted in [22]: these models account for the correlation between consecutive inter-occurrence times. In this paper, we shall evaluate the above-mentioned approximation approaches when the model parameters are estimated from the pooled output.

The contribution of the article is multi-fold. First, an innovative procedure to evaluate the performance of five different approximation approaches is presented based on density level sets. Second, the estimation procedures implemented for each approximating model of an SRP are derived and compared. The third contribution consists of taking into account periodic preventive maintenance within the pooled outputs. Condition-based maintenance requires more complex modeling and is beyond the scope of the paper. Finally, we emphasize the selection of particular approximation methods along with specific inference procedures on a real data set from Norwegian railway signaling systems.

The rest of the paper is structured as follows. In Section 2, the SRP notations are introduced before presenting a new performance-measuring procedure based on level-set. Section 3 addresses the inference procedures and evaluates the performance of the approximations. An application to real data is proposed in Section 4. Most systems in the Norwegian railway signaling network operated by Bane NOR can be assumed to be series systems under partial repair with both corrective and potential preventive maintenances. The benefits of the aforementioned approximations are highlighted through the modeling and estimation procedures based on the pooled output of signaling systems and relevant prognosis indicators.

2. APPROXIMATION OF A SRP

2.1. Notation

Consider a maintainable series system under partial repair. A corrective maintenance (CM) is carried out immediately after a failure and consists of replacing the failed component while maintaining the others minimally. Periodic preventive maintenance (PM) is performed at $\Delta, 2\Delta, 3\Delta, \dots$, and $\Delta = \infty$ if no PM is implemented. It is also assumed that at $t = 0$, the system is in its stationary regime, that is, has been functioning for a long while. The

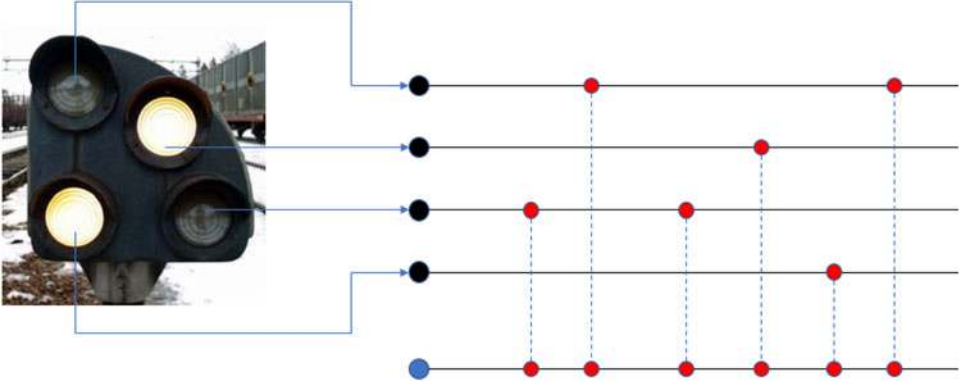


FIGURE 1. Illustrative trajectory of a CM process for a dwarf signal.

identity of the components causing the failures is assumed to be unknown. Observations can be modeled by a colored point process: failure times are denoted by $\{T_i\}_{i \geq 1}$, with inter-arrival times $\{X_i\}_{i \geq 1}$. The indicator of the maintenance types are $\{\delta_i\}_{i \geq 1}$ with $\delta_i = 0$ for CM and $\delta_i = 1$ for PM. Figure 1 presents an hypothetical trajectory of a maintenance process for a dwarf signal, wherein the lamps are in series configuration: the four lights function independently and are subject only to CM, and no identification regarding the renewed bulbs is available. It is, therefore, reasonable to model the pooled failure times $\{T_i\}_{i \geq 1}$ by an SRP.

2.2. Reminder of the Current SRP Approximations

SRP are usually approximated by a standard renewal process [33,34]. In a first study [22], we have highlighted the benefits of using virtual ages or copulas to approximate an SRP:

- A virtual age model [18] is an imperfect maintenance model which assumes that after maintenance at time T_i , the system behaves as a new and unmaintained one of age A_i . The virtual age model that approximates plausibly an SRP when components are homogeneous is the arithmetic reduction of age with infinite memory, ARA_∞ [11,25], which assumes that the virtual age of the system after a maintenance is reduced proportionally:

$$A_i = (1 - \rho)(A_{i-1} + X_i), \quad \rho \in [0, 1]. \quad (1)$$

ARA_∞ is a stable process [23] in the sense that its intervals converge in distribution. Like in an SRP, the failures in an ARA_∞ model arrive at a constant rate, asymptotically.

- A copula model [16,30] is a convenient approach to characterize a joint distribution by defining the marginal distributions and a copula function. The copula associated with the observation of two consecutive inter-arrival times of an SRP in its steady regime (X_i, X_{i+1}) has been employed to approximate an SRP. This modeling is automatically more efficient than a renewal process as it characterize the distribution of X_{i+1} given X_i . This model is denoted by inter-arrival-time copula of memory 1 (IAT_1).

When the parameters of the approximated SRP are known, ARA_∞ and IAT_1 outperform the renewal process approximation for giving more accurate estimation of the

remaining useful life (RUL) [22], with the ARA_∞ being marginally better than IAT_1 . In addition, both ARA_∞ and IAT_1 capture the negative dependency between two consecutive inter-arrival times when all the components have an increasing failure rate (IFR) [22]. The correspondence between SRP and ARA_∞ can be emphasized further using one of the strongest measures of negative dependence called reverse regular of order 2 (RR_2) [1]. The proof is presented in [Appendix A](#).

PROPOSITION 2.1: *Two successive intervals in an SRP, formed by components with increasing failure rates, are RR_2 -dependent. Similarly, two successive inter-arrival times of an ARA_∞ process with increasing baseline failure intensity are RR_2 -dependent.*

2.3. A Level-Set Procedure to Measure the Performance of the Models

Let us consider a risk α and a trajectory of SRP observations where the parameters of the model and the age of each component are available. At time T_i , it is straightforward to compute the confidence interval of smallest amplitude $I = [T_i + a, T_i + b]$ with confidence level $100(1 - \alpha)\%$ from the actual model. Next, an interval $J = [T_i + a', T_i + a' + b - a]$ of amplitude $b - a$ can be determined which maximizes the confidence set of the approximating model. The probability that T_{i+1} belongs to I is naturally $1 - \alpha$, and the probability that T_{i+1} belongs to J is automatically less than $1 - \alpha$. The quality of the approximating model lies in how the latter probability is close to $1 - \alpha$. This probability p_α can be estimated empirically from a unique and sufficiently long trajectory by assessing the proportion of times when the prediction has been correct. Furthermore, the overall performance of the approximated model can be synthesized by its Gini index $G = 2 \int_0^1 (1 - \alpha - p_\alpha) d\alpha$. If $G = 0$, the approximating model is indistinguishable from the actual model, and $G = 1$ is the worst-case scenario where the approximating model is totally inconsistent with SRP.

Five approximated models are considered, and their parameters are assumed to be known in this section only:

- HPP.** The homogeneous Poisson process (HPP) with a rate is equal to the asymptotic rate of the SRP. This model is the simplest SRP approximation and is valid when the number of sources in the SRP tends to infinity [12].
- SIM.** The stationary interval method (SIM; [33]) corresponds to a renewal process with a generic distribution equal to the asymptotic distribution of the SRP.
- IAT_1 .** Inter-arrival-time copula of memory 1, as reminded in Section 2.2.
- ARA_∞ .** ARA_∞ process, as reminded in Section 2.2.
- BP.** The Brown–Proschan model (BP; [4]). Belonging to imperfect maintenance models, BP assumes that maintenance is either perfect with probability p or imperfect with probability $1 - p$. Like an SRP, BP has a stationary regime [10]. It is conceivable that SRP with heterogeneous components are relatively better approximated by BP, wherein the maintenance is heterogeneous. The parameters chosen for the BP model are the shape and scale parameters of the initial Weibull failure rate and the parameter p which has the best goodness-of-fit (GOF) with the initial SRP.

Five SRP configurations are studied as follows:

- I. A system formed of three homogeneous components with a low wear-out rate. The failure rate of each component follows a Weibull distribution $\mathcal{W}(\eta, \beta)$ with a shape parameter $\beta = 1.5$ and a scale parameter $\eta = 1$.

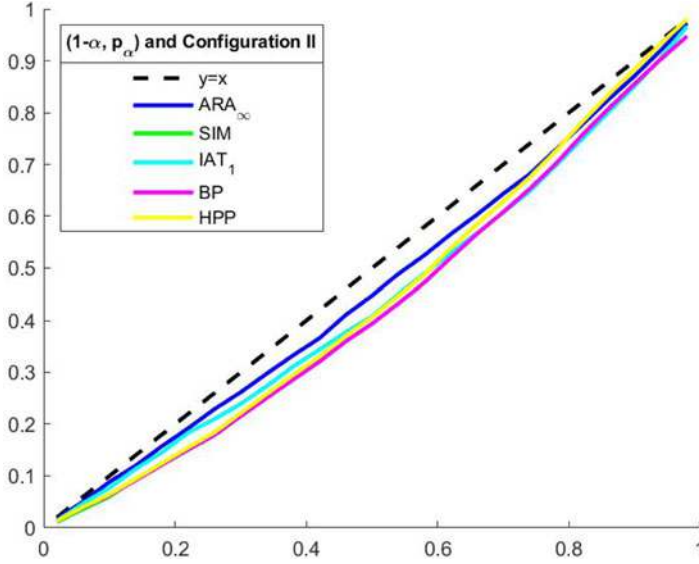


FIGURE 2. Level sets in the fast aging rate cases: homogeneous components (configuration II).

- II. A system formed of three homogeneous components with a fast wear-out rate where each component follows a Weibull distribution $\mathcal{W}(\eta = 1, \beta = 3.5)$.
- III. A system formed of three heterogeneous components with a low wear-out rate. The failure rate of each component follows a Weibull distribution with $\beta = 1.5$ and respective scale parameters 1, 2 and 10.
- IV. A system formed of three heterogeneous components with a fast wear-out rate. The failure rate of each component follows a Weibull distribution with $\beta = 3.5$ and respective scale parameters 1, 2 and 10.
- V. A system formed of six relatively homogeneous components with a moderate wear-out rate. The components have a Weibull lifetime distributions: three are $\mathcal{W}(\eta = 1, \beta = 2.5)$ and three are $\mathcal{W}(\eta = 2, \beta = 2.5)$.

For each model and each configuration, the level set p_α is computed and compared to the initial confidence level $100(1 - \alpha)\%$ through the identity line. Figures 2 and 3 present the results with a fast wear-out for the configurations II and IV, respectively. The figures for other configurations are not presented as the plots are almost indistinguishable from the identity line. In addition, each Gini coefficient is provided in Table 1.

The analysis of the level sets and Gini index completes and refines the initial analysis from [22]:

- For each model, the Gini index is relatively low, between 0.002 and 0.2, which indicates that each model provides an efficient approximation of an SRP. The Gini index are particularly low when the aging of the components are moderate or when the number of components is important.

The model under ARA_∞ assumption almost consistently provides the best index, regardless of the heterogeneity of the components and the aging rates of the components.

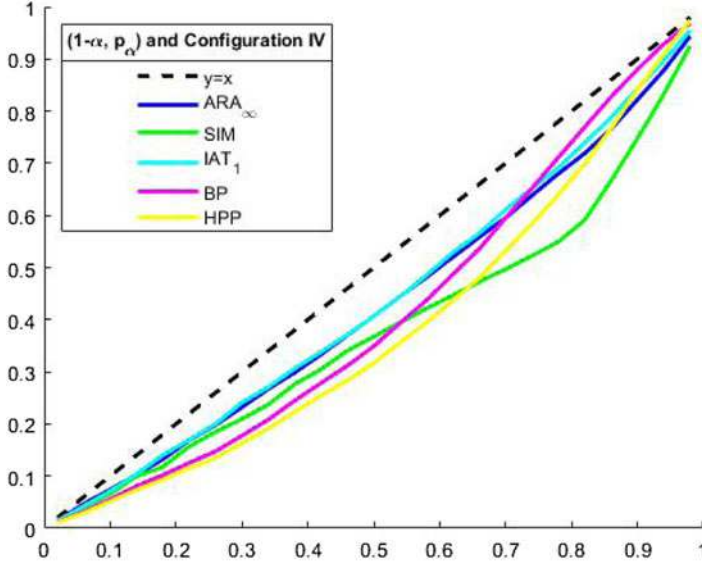


FIGURE 3. Level sets in the fast aging rate cases: heterogeneous components (configuration IV).

TABLE 1. Gini coefficients (10^{-3})

	HPP	SIM	IAT ₁	ARA _∞	BP
I	9.1	8.9	8.5	8.2	8.6
II	119.2	119.5	137.0	89.4	144.0
III	6.5	7.2	6.2	3.0	5.9
IV	222.7	230.1	116.1	133.8	204.5
V	2.8	4.3	4.3	3.0	5.6

- The IAT₁ and BP models present satisfactory but more contrasting results. The BP approximation is comparatively much less efficient when the components are aging fast (configurations II and IV). The IAT₁ approximation outperforms the other models in the heterogeneous case with fast aging rates (configuration IV) but provides, otherwise, similar results to the HPP approximations.
- The two static models HPP and SIM presents decent and very similar results. When the aging rate is moderate, their efficiency has the same order of magnitude as the other adaptive models, but they are clearly outperformed by ARA_∞ when the wear-out rate is fast.
- All the models are extremely efficient when the number of components is large, particularly the HPP which is the theoretical limiting case when the number of components tends to infinity [12].

3. PERFORMANCE OF THE APPROXIMATIONS WITH UNKNOWN PARAMETERS

The parameters of the models are commonly not known and need to be estimated first from a relatively small data set. The performance of the inference procedures might put the previous analysis into perspective.

3.1. Configurations

The performances of the SRP approximations are discussed for the five models: HPP, SIM, IAT_1 , ARA_∞ and BP. The number of observations is assumed to be larger than 10 without exceeding 200, and the system is assumed to be in its stationary regime at the beginning of the observation. Three configurations previously studied in Section 2.3 are considered, two with CM only and one with CM and periodic PM:

- (A) The configuration II with three homogeneous components and a relatively fast wear-out ($\mathcal{W}(\eta = 1, \beta = 3.5)$) and CM only.
- (B) The configuration IV with three heterogeneous components and a relatively fast wear-out ($\mathcal{W}(\eta_i = \{1, 2, 10\}, \beta = 3.5)$) and CM only.
- (C) The configuration II with three homogeneous components and a relatively fast wear-out with both CM and PM. The periodicity of the preventive maintenance policy is $\Delta = 0.425$. The periodicity has been chosen so that the proportion of PM in the actual SRP model is 70%. The maintenance efficiency is assumed to be imperfect but of great quality on each component. Therefore, it has been opted to assume that PM efficiencies are ARA_∞ with an improvement factor $\rho = 0.7$. The virtual age of each component after a preventive maintenance action is 30% of its virtual age just before the PM.

These three configurations are not exhaustive but are quite representative of the behavior of an SRP with or without homogeneity. The PM policy in (C), which corresponds to the case where imperfect PM are carried out simultaneously on all the components, is investigated because it is easy to implement and widely applied in practice.

3.2. Measures of Performance

The objective is to quantify the efficiency of the approximation approaches when the number of observations is limited. Two predictors, the mean lifetime $\mathbb{E}[X_\infty]$ and the Pearson's correlation coefficient Corr between two successive inter-failure times, have been considered. They are assumed to be known for the SRP models, either computed theoretically considering CM only or obtained based on Monte Carlo simulations at any given precision when both CM and periodic PM are implemented. Given a sample size N of an SRP configuration, the mean squared error (MSE) of the two predictors $\mathbb{E}[X_\infty]$ and Corr are derived empirically for the five approximated models based on 5,000 replicates of histories. A general structure of the simulation procedure is presented in Algorithm 1.

3.3. Inference Procedures

Additional information is provided on the estimation methods for the five models corresponding to lines 7 and 8 in Algorithm 1. In the case of no PM, the observations consist of the inter-failure times $\{X_i\}_{i=1\dots N}$ or equivalently the maintenance times $\{T_i\}_{i=1\dots N}$. In the case of both PM and CM, the observations additionally comprise the maintenance types $\{\delta_i\}_{i=1\dots N}$.

3.3.1. Homogeneous Poisson process The failure rate λ in an HPP can be assessed by MLE:

$$\hat{\lambda} = \frac{n - \sum_{i=1}^n \delta_i}{T_n}. \quad (2)$$

Algorithm 1 Empirical MSE computation

```

1: Select a SRP configuration
2: Compute the predictors  $e = \mathbb{E}[X_\infty]$  and  $r = \text{Corr}$ 
3: for  $N \in \{10, 20, 50, 100, 200\}$  //  $N = \text{length of a trajectory}$ 
4:   for  $k = 1 : 5000$  //  $k = \text{th sample}$ 
5:     Simulate a SRP trajectory Hist of length  $N$ 
6:     for  $i = 1 : 5$  //  $i = \text{th model (HPP, SIM, IAT}_1, \text{ARA}_\infty, \text{BP)}$ 
7:       Assess the parameters of the  $i$ th model from Hist
8:       Estimate the predictors by  $\widehat{e}_i(k)$  and  $\widehat{r}_i(k)$ 
9:     end for
10:   end for
11:   for  $i = 1 : 5$ 
12:     Compute the empirical MSE of each measure for a trajectory of
length  $N$  and the approximated model  $i$ :
13:      $MSE_{\widehat{E}}(i, N) = 1/5000 \sum_{k=1}^{5000} (\widehat{e}_i(k) - e)^2$ 
14:      $MSE_{\widehat{C}_{\text{Corr}}}(i, N) = 1/5000 \sum_{k=1}^{5000} (\widehat{r}_i(k) - r)^2$ 
15:   end for
16: end for

```

The first predictor, $\mathbb{E}[X_\infty]$, equals $1/\widehat{\lambda}$. The second predictor, Corr , is 0 since the intervals are independent in an HPP.

3.3.2. Stationary interval method The SIM method approximates an SRP by a renewal process whose interval is distributed identically as X_∞ . This distribution can be roughly estimated by the empirical survival function in case of CM only or by the Kaplan–Meier estimator function in the presence of CM and PM.

To evaluate the accuracy of SIM approximation, $\mathbb{E}[X_\infty]$ is computed by numerically integrating a piecewise-linear version of the estimated survival function, and the correlation is 0 given that successive inter-arrival times are independent for a renewal process. Within the course of the inference procedures, additional non-parametric estimations of the reliability function have been tested using different kernels and several bandwidths, but the piecewise-linear version of the Kaplan–Meier has been opted as it consistently provided the best results.

In the presence of PM, the Kaplan–Meier estimator is not defined beyond the largest observation if the longest interval is censored, which makes it impossible to compute the expected value. We adopt therefore the common approach, that is change the largest observation to a death time if it is censored [13].

3.3.3. Inter-arrival-time copula of memory 1 The IAT_1 is a refined version of SIM: it takes into account additionally the correlation between two successive intervals in the stationary regime by defining a copula. A semi-parametric approach is carried out, wherein the marginal survival functions is assessed in the same way as the SIM method using a Kaplan–Meier estimator, and the copula function is estimated from the Frank copula family, characterized by a single parameter θ as shown in Eq. (3). Both negative dependence ($\theta < 0$), positive dependence ($\theta > 0$) and independence ($\theta = 0$) can be modeled within the Frank

family.

$$C_\theta(u, v) = -\frac{1}{\theta} \log \left(1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1} \right). \quad (3)$$

The choice of the Frank family is explained as follows. The configurations of the SRP defined in Section 2.3 have been studied. After generating an SRP sequence of length $N = 500,000$, several copula families (Gaussian, t , Frank, Gumbel, Clayton and independent copula) are then fitted to the data. It has been found that there exists a certain convergence between the families:

- The degree of freedom ν in the t -copula is systematically estimated to be extremely large, corresponding to the Gaussian copula. Therefore, only the Gaussian copula is maintained in the comparative study
- The parameter of the Gumbel family, $\hat{\theta}_{\text{Gumbel}}$ is evaluated to be 1 and that of the Clayton family, $\hat{\theta}_{\text{Clayton}}$ is estimated to be 0_+ , which corresponds in both cases to the independent copula. As the Gumbel and Clayton copulas do not take into account the negative dependency, both models are not kept for the comparative study.

Since the correlation between the intervals is of interest to the current paper, the independent copula, as well as the Gumbel and Clayton, are excluded from consideration. Gaussian and Frank copula are now the two candidates. In the following, we investigate whether they differ in estimating the tail dependence.

The concept of tail dependence describes the amount of dependence in the lower-left-quadrant tail or the upper-right-quadrant tail of a bivariate distribution. Let F_1 and F_2 be the marginal distributions of variables X_1 and X_2 , respectively. A common measure of tail dependence is given by the so-called upper/lower tail dependence function [14]:

$$\lambda_L(v) = P(F_1(X_1) \leq v | F_2(X_2) \leq v) = \frac{C(v, v)}{v}, \quad (4)$$

$$\lambda_U(v) = P(F_1(X_1) \geq v | F_2(X_2) \geq v) = \frac{1 - 2v + C(v, v)}{1 - v}. \quad (5)$$

An example of the tail dependence functions $\lambda_L(v)$, $v \in [0, 0.5]$ and $\lambda_U(v)$, $v \in [0.5, 1]$ are shown, respectively, in Figures 4 and 5 for the SRP configuration II. The tail dependence of the intervals in SRP is estimated empirically [5]. It is observed that the Frank (triangle marker) and the Gaussian copulas (circle marker) are both close to the empirical SRP (cross marker). The independent copula is clearly inadequate.

The GOF tests [15] are then used to assess to what extent the copula in an SRP resembles a Frank copula or a Gaussian copula. The copula under null hypothesis is Frank or Gaussian. Let α be the level of significance. With infinite data, an SRP copula can never be approximated by a Frank/Gaussian copula, meaning that the null hypothesis will be 100 percent rejected (p -value less or equal to α). When the amount of observations are limited, however, the SRP copula can be estimated very satisfyingly by a Frank/Gaussian copula.

Let $\alpha = 0.05$. An SRP sequence of length $N = 20$ is generated 1,000 times, and the corresponding copula (which describes the dependence structure between two successive intervals in an SRP) is tested against Frank/Gaussian, using the Cramer–von Mises statistics [15]. The total proportion of an SRP copula being classified as Frank/Gaussian, is recorded in Table 2, column “GOF Frank” and “GOF Gaussian.” It is found that the proportion of SRP copula, which “look like” a Gaussian copula, is slightly higher than that for a Frank copula.

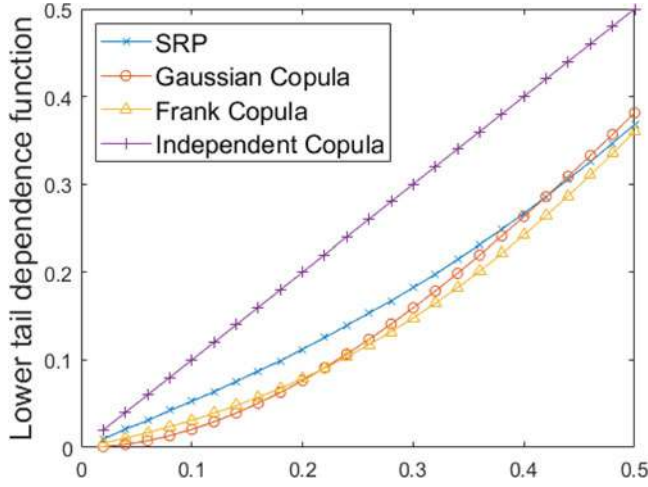


FIGURE 4. Lower tail dependence function.

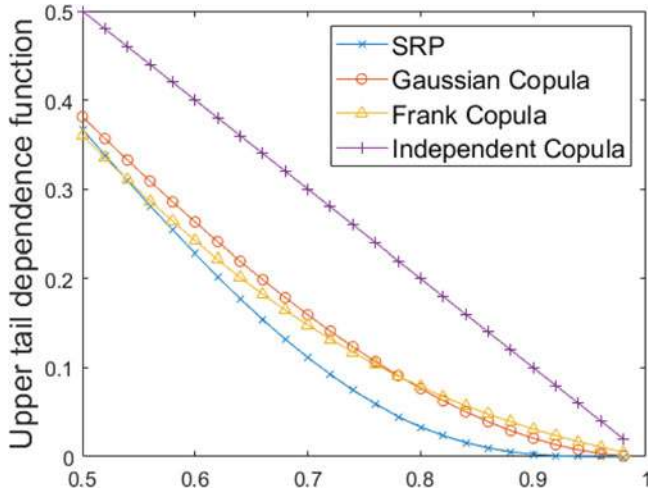


FIGURE 5. Upper tail dependence function.

TABLE 2. Estimated copula parameters and results of test of GOF

SRP	θ in Frank copula	ρ in Gaussian copula	GOF Frank	GOF Gaussian
I	-0.7372	-0.1234	0.949	0.966
II	-2.3652	-0.3634	0.926	0.952
III	-0.6137	-0.1046	0.955	0.963
IV	-2.0441	-0.3151	0.926	0.946

To sum up, both Frank and Gaussian copulas have similar performance in estimating the tail dependence. In terms of the GOF test, Gaussian is slightly better than Frank. Nevertheless, Frank copula admits an explicit formula which is not possible for Gaussian. This facilitates greatly the inference procedure when missing/censored data is involved, as shown later in the case studies.

Once the marginal functions have been characterized, θ can be estimated by maximum likelihood estimation, and the two predictors can be assessed. In the presence of both CM and PM, the estimation procedure is more complex as it potentially involves doubly censored bivariate data. This issue has been specifically addressed in [29].

3.3.4. Arithmetic reduction of age with infinite memory Using imperfect maintenance models such as the ARA_∞ virtual age assumption is an alternate possibility to approximate an SRP. Inference procedures have already been implemented for virtual age models when the system, under CM only, is assumed to be in its stationary regime at the beginning of the observations [9] for any given initial intensity. In the following, it is assumed that the initial intensity follows a Weibull distribution, which is a common assumption for aging systems in reliability engineering. For the configurations A and B, three parameters need to be estimated: η and β , respectively, the scale and shape parameter of the initial Weibull intensity and $\rho_0 \in]0, 1]$, which characterizes the maintenance efficiency. In the presence of CM and PM such as in configuration C, it is also assumed that the PM efficiency follows an ARA_∞ assumption with a parameter ρ_1 , with traditionally $\rho_1 > \rho_0$. The parameters of the models can be estimated by maximum likelihood estimation in a similar way as in [9]. Let us denote $\lambda(t) = (\beta/\eta)(t/\eta)^{\beta-1}$ the initial Weibull intensity and $\Lambda(t) = (t/\eta)^\beta$ the cumulative intensity. First, the likelihood function \mathcal{L}_{a_0} associated with the observation of the n first maintenance times and types $(\mathbf{X}, \boldsymbol{\delta})_{1\dots n}$ and an initial age a_0 can be expressed as in Eq. (6). The effective ages in Eq. (6) can be obtained by induction given a_0 and using $a_i = (1 - \rho_{\delta_i})(a_{i-1} + x_i)$.

$$\mathcal{L}_{a_0}((\mathbf{X}, \boldsymbol{\delta})_{1\dots n}) = \prod_{i=1}^n (\lambda(a_{i-1} + x_i))^{1-\delta_i} \times \exp\left(\sum_{i=1}^n -\Lambda(a_{i-1} + x_i) + \Lambda(a_{i-1})\right). \quad (6)$$

Second, as the SRP is assumed to be in its stationary regime at the beginning of the observations, the same assumption is proposed for its approximating model. It implies that a_0 is the realization of the limiting age distribution A_∞^{ARA} with pdf $f_{A_\infty^{\text{ARA}}}$. The resulting likelihood function \mathcal{L} associated with the observation of the n first maintenance times and types $(\mathbf{X}, \boldsymbol{\delta})_{1\dots n}$ and an initial mixing distribution A_∞^{ARA} is presented in Eq. (7). The mixing distribution A_∞^{ARA} has been characterized theoretically in Nguyen *et al.* [25] for CM only and can be derived empirically from intensive simulations considering CM and PM.

$$\mathcal{L}((\mathbf{X}, \boldsymbol{\delta})_{1\dots n}) = \int_0^\infty \mathcal{L}_a((\mathbf{X}, \boldsymbol{\delta})_{1\dots n}) f_{A_\infty^{\text{ARA}}}(a) da. \quad (7)$$

Finally, once the parameters of the model estimated, $\mathbb{E}[X_\infty]$ and Corr can be computed numerically, as shown in Nguyen *et al.* [25] and Liu *et al.* [22], respectively.

3.3.5. Brown–Proschan model The BP virtual age assumption is another candidate to approximate an SRP as both converge to a stationary regime. Consider an initial Weibull intensity. Estimation methods have been presented in Lim [20], Doyen [10] and specifically in the case of initial steady-state in Dijoux *et al.* [9] for CM only. With only CM, the parameters to estimate are the scale and shape parameters of the Weibull distribution and the CM efficiency $p_0 \in]0, 1]$. With both CM and PM, the PM efficiency is assumed to follow a BP assumption with a parameter p_1 . The likelihood function associated with the observation of n maintenance times and types can be derived in three steps. First, the likelihood function $\mathcal{L}_{a_0}^{\text{NHPP}}$ associated with the observation of a system with initial age a_0

under minimal repair are reminded in Eq. (8) using the classic results of non-homogeneous Poisson processes (NHPP) [28].

$$\mathcal{L}_{a_0}^{\text{NHPP}}((\mathbf{X}, \boldsymbol{\delta})_{1\dots n}) = \prod_{i=1}^n (\lambda(a_0 + t_i))^{1-\delta_i} \times \exp(-\Lambda(a_0 + t_n) + \Lambda(a_0)). \quad (8)$$

Second, the likelihood function $\mathcal{L}_{a_0}^{\text{BP}}$ associated with the observation of a system with initial age a_0 and BP efficiencies are derived recursively in Eq. (9) using a similar approach as in [9].

$$\begin{aligned} \mathcal{L}_{a_0}^{\text{BP}}((\mathbf{X}, \boldsymbol{\delta})_{1\dots n}) &= \left[\prod_{k=1}^{n-1} (1 - p_{\delta_k}) \right] \times \mathcal{L}_{a_0}^{\text{NHPP}}((\mathbf{X}, \boldsymbol{\delta})_{1\dots n}) \\ &+ \sum_{i=1}^{n-1} \mathcal{L}_{a_0}^{\text{BP}}((\mathbf{X}, \boldsymbol{\delta})_{1\dots i}) p_{\delta_i} \left[\prod_{k=i+1}^{n-1} (1 - p_{\delta_k}) \right] \times \mathcal{L}_0^{\text{NHPP}}((\mathbf{X}, \boldsymbol{\delta})_{i+1\dots n}). \end{aligned} \quad (9)$$

Finally, the initial age a_0 is traditionally not known and can be assumed to be the realization of a random variable A_∞^{BP} , limiting distribution of the effective age in the stationary regime. The resulting likelihood function \mathcal{L}^{BP} associated with the observation of the n first maintenance times and types $(\mathbf{X}, \boldsymbol{\delta})_{1\dots n}$ and an initial mixing distribution A_∞^{BP} is presented in Eq. (10).

$$\mathcal{L}^{\text{BP}}((\mathbf{X}, \boldsymbol{\delta})_{1\dots n}) = \int_0^\infty \mathcal{L}_a^{\text{BP}}((\mathbf{X}, \boldsymbol{\delta})_{1\dots n}) f_{A_\infty^{\text{BP}}}(a) da. \quad (10)$$

The distribution of A_∞^{BP} has been expressed theoretically in [10] with CM only. With CM and PM, the pdf of A_∞^{BP} can be obtained numerically. Once the parameters of the models are estimated, the predictors can be obtained from a plugged-in version of their theoretical expressions or numerically.

3.4. Results

The MSE of the two predictors $\mathbb{E}[X_\infty]$ and Corr are, respectively, plotted in Figures 6 and 7 for configuration A, Figures 8 and 9 for configuration B and Figures 10 and 11 for configuration C.

When no PM is involved (configurations A and B), HPP has the smallest error in the estimation of $\mathbb{E}[X_\infty]$ (see Figures 6 and 8). Virtual age models (ARA $_\infty$ and BP) perform better than IAT $_1$ and SIM. With more than 50 data, the differences between the MSEs given by the tested models are less important.

As for the error of the predictor Corr (see Figures 7 and 9), HPP and SIM have the largest error since their intervals are independent. BP is an auto-correlated process with a weak dependence between intervals, but its estimation of correlation is far from accurate. IAT $_1$ has the smallest MSE when the SRP are homogeneous, which may support the practice of using a Frank copula to approximate the dependence structure in such an SRP. If the SRP differ strongly from each other as in configuration B, ARA $_\infty$ outperforms other models in estimating the correlation between adjacent intervals.

When periodic PM are implemented (Figures 10 and 11), the SIM and IAT $_1$ have a considerable error in $\mathbb{E}[X_\infty]$. HPP performs the best when estimating $\mathbb{E}[X_\infty]$, but with more data, the advantage of HPP over BP/ARA $_\infty$ is less significant. As for the correlation estimator, the performance of the tested models depend on the data length: with less than 50 data, HPP outperforms the others; otherwise, BP and ARA $_\infty$ are the best. It should

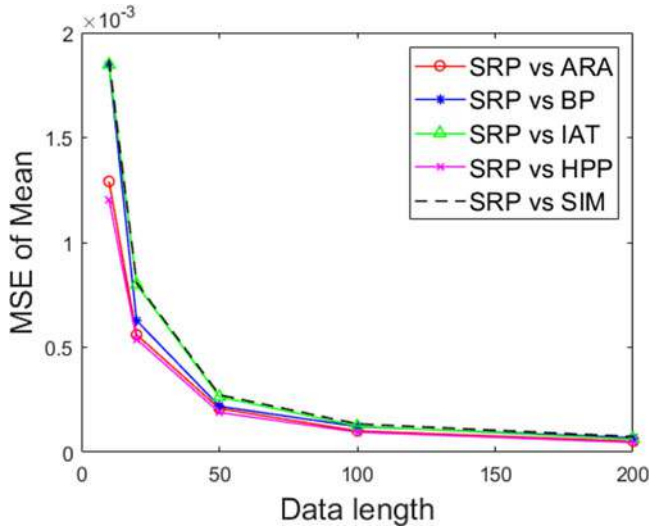


FIGURE 6. Configuration A: MSE of $\mathbb{E}[X_\infty]$.

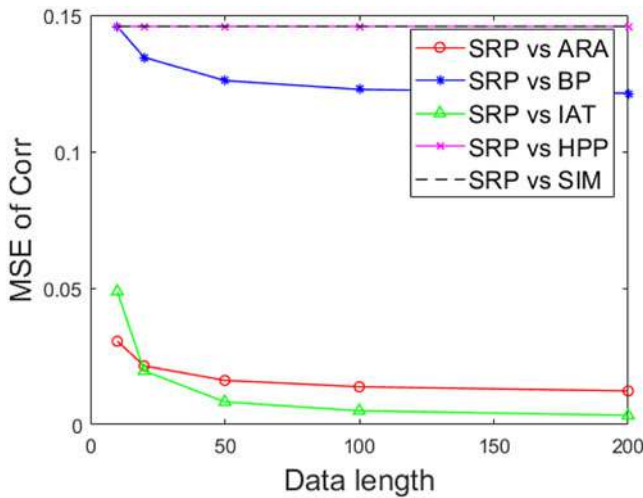


FIGURE 7. Configuration A: MSE of Corr.

be emphasized that the implementation of periodic PM results in a non-null correlation in HPP/SIM sequences.

4. A BANE NOR CASE STUDY

4.1. General Overview and Data Presentation

Signaling is essentially a sophisticated traffic light system for the railway. The complexities of moving trains around such a large network, keeping them safely apart, and allowing for their long stopping distances, means that the signaling system is very complicated and comprises a great many parts.

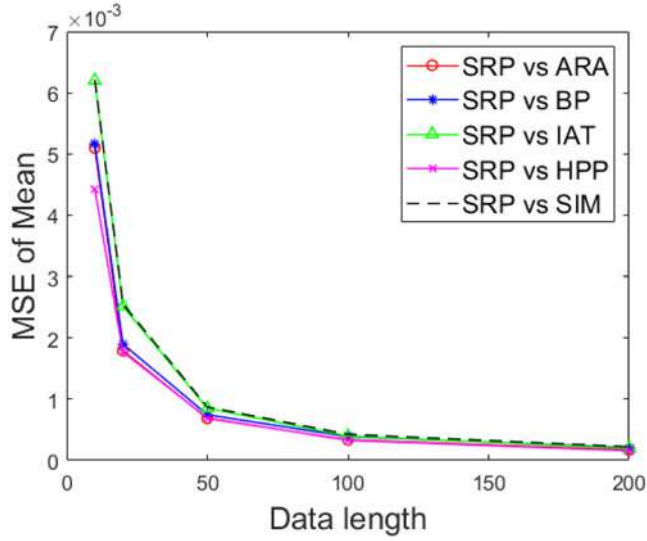
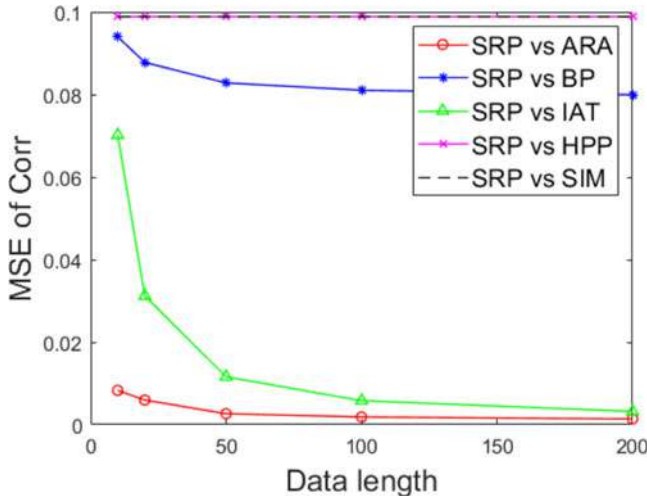
FIGURE 8. Configuration B: MSE of $\mathbb{E}[X_\infty]$.

FIGURE 9. Configuration B: MSE of Corr.

The signals themselves are the line-side pieces of equipment that tell train drivers when it is safe to proceed and what route their train will take. A light signal comprises (1) signal head with light sources, background screen and shadow screens; (2) mast with platform/ladder when needed and (3) devices for controlling the signal with interface to interlocking equipment.

In this section, we focus on the sub-system of the light sources in a signal head, formed by two to five lights of different colors (red, green, yellow and white). One lamp can be either LED or incandescent. *A priori*, there is no manifest heterogeneity in the lights of the same type, but the maintenance strategy depends on the importance of the lamp: since the failure of a red light which gives the signal “stop” is often more severe than a failure of a green one, preventive maintenances have been performed only on the signals that consists of

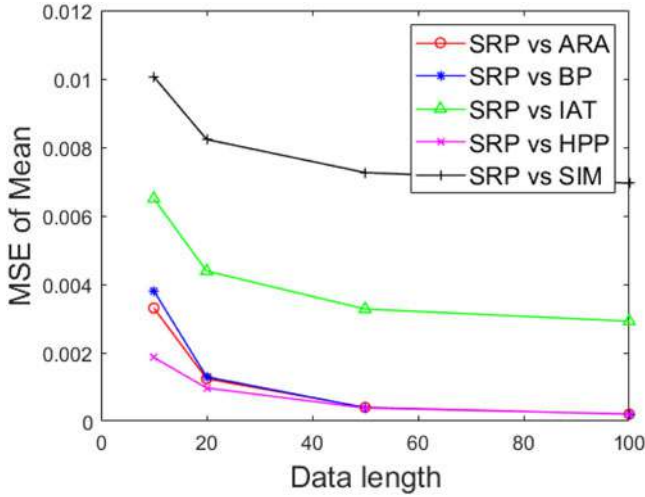


FIGURE 10. Configuration C: MSE of $E[X_\infty]$.

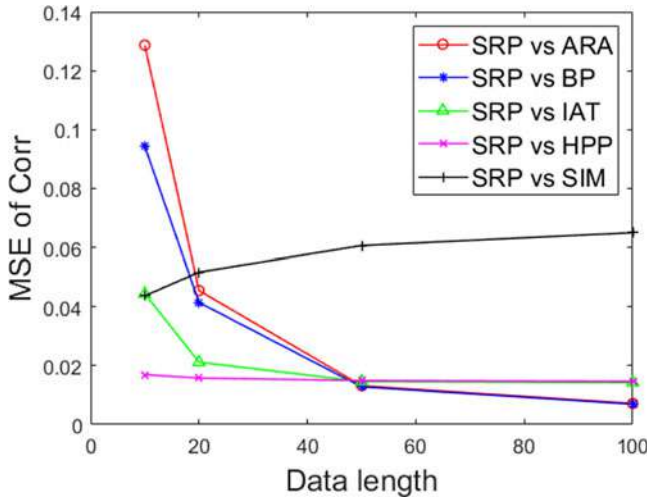


FIGURE 11. Configuration C: MSE of Corr.

red light bulbs. Both corrective and preventive maintenances consist in replacing the light bulb, burned or still working, by a new one. Some preventive maintenances, like periodic inspection or cleaning, are planned for other parts of the signal (cables, covering glass, etc.), and are not considered here.

However, the information regarding the maintenance actions is incomplete. When a burned light bulb is replaced, its position in the system is usually not recorded. Thus, the failure history of an individual light bulb is not available and the failures of the system must be treated as a whole. An SRP can, therefore, be used to describe the successive failures of the light sources in a signal.

In the following, three case studies are presented where the proposed five approximation models are fitted to the failures times of dwarf signals. The first study is on a single asset, showing how to use these models to compute the expectation of RUL and evaluating the

TABLE 3. A summary of the case studies

	Object	Inference	Censoring	No. asset
Case study 1	Asset 012110	Individual	No	1
Case study 2	Signals having 7 ⁺ failures	Grouped	Yes	16
Case study 3	All dwarf signals	Grouped	Yes	1608

TABLE 4. Inter-failure times of asset 012110

133	10	283	763	19	378	203	920
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TABLE 5. Parameters of the models fitted to asset 012110

	Parameters	A_0	$\mathbb{E}[X]$	Expected value of 9th lifetime
ARA _∞	$\alpha = 3.34 \times 10^{-10}$			
	$\beta = 2.9737$	1717.1	321.38	239.83
	$\rho = 0.1577$			
BP	$\alpha = 3.27 \times 10^{-10}$			
	$\beta = 3.3571$	460.77	321.91	293.36
	$p = 0.4113$			
HPP	339.08	0	339.08	339.08
IAT ₁	$\theta = -1.32$	0	281.52	210.49
SIM		0	281.52	281.52

virtual age; the second one focuses on signals having numerous failure records (more than 7), with the assumption that despite the geographical heterogeneity, these signals could be described by models of the same parameters; the third one studies all the dwarf signals composed by four incandescent light without redundancy (single filament lamp).

The maintenance records provided by Bane NOR contains not only failure histories (CM) but also some invalid PM plans. The earliest date that appeared in a PM record is considered as the beginning of the individual observation window. For the first case studies, the observations start from the first recorded CM; in the second and third case study, the beginning of observation is the earliest date of the planned PM, determined individually for each asset; and the end of observation is the last date of CM, which is March 28, 2019. Thus, for each asset, there is at least one right-censored data (Table 3).

4.2. Case Study 1: Asset 012110

The investigated signal is located in Dovre Line, a main national connection between Eastern Norway and Trøndelag and further north for passenger and freight traffic. The consecutive lifetimes are shown in Table 4.

Parameters of fitted models and some reliability indicators are gathered in Table 5. The second column A_0 represents the expected virtual age at the beginning of the observation and needs to be calculated only for imperfect maintenance models, namely ARA_∞ and BP. $\mathbb{E}[X]$ is the unconditional expected lifetime in each model. For the models which are not renewal processes, the expected value of the 9th lifetime is smaller than $\mathbb{E}[X]$ since the last observed lifetime is relatively large.

The calculation of conditional $\mathbb{E}[X]$ under ARA_∞ and BP is recalled here. In ARA_∞, A_0 is derived along with the parameters, then the consecutive effective virtual ages are

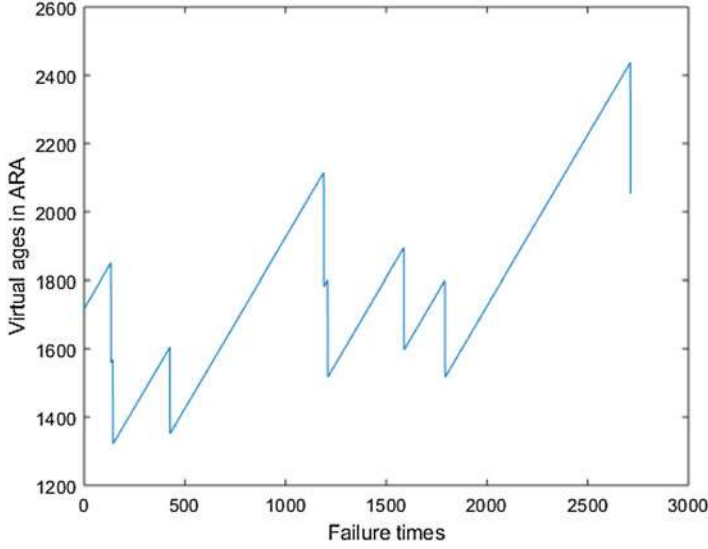


FIGURE 12. Virtual ages in the approximating ARA_∞ model.

calculated iteratively by Eq. (1) and plotted in Figure 12. Denote by A_{end} the virtual age after last observation: the conditional reliability of the next interval is $R_{ARA_\infty}(t|A_{\text{end}}) = \exp(-\alpha(t + A_{\text{end}})^\beta + \alpha A_{\text{end}}^\beta)$.

As for BP, one should first retrieve the maintenance effects, which is described in [20]. The repair effect of the 1st–7th maintenance actions can be represented by a binary vector $B = [0, 0, 1, 0, 1, 0, 1]$, where 0 signifies a minimal repair and 1 a perfect one. Remark that the maintenance effect after the last CM is non-identifiable in the absence of further data. A_0 is calculated [10] and the consecutive virtual ages are plotted in Figure 13. After the 8th repair, the system is assumed to be as good as new. The repair after the last observed lifetime has the probability p_0 to be perfect and $1 - p_0$ to be minimal. Thus, the reliability of the 9th interval can be expressed as $R_{BP}(t|B) = p \cdot \exp(-\alpha t^\beta) + (1 - p) \cdot \exp(-\alpha(t + X_{\text{end}})^\beta + \alpha X_{\text{end}}^\beta)$, where X_{end} is the last observed lifetime.

It can be observed from Figures 12 and 13 that the asset seems to be older under ARA_∞ assumption than it is under the BP model. This is because (1) a low maintenance efficiency was estimated for ARA_∞ , which limits the reduction of age at each repair and (2) three maintenances are estimated to be perfect in the BP model, making the asset relatively young. Consequently, the remaining lifetime predicted by ARA_∞ is smaller than that evaluated by BP (Figure 14).

4.3. Case Study 2: the Most Frequently Failed Assets

We investigate the inference and the hazard rate in the 16 most frequently failed assets. One may assume that there exist some clusters of assets in which all members are considered similar and share the same parameters of a certain model. The clustering can be based either on physical/geometrical features, that is, assets located at the same station and composed by lamps of the same type, or on lifetimes features, that is, a bunch of signals whose MTBF are significantly different from the others. The inference procedure for clustered assets is to maximize the sum of the log-likelihood function of all group members.

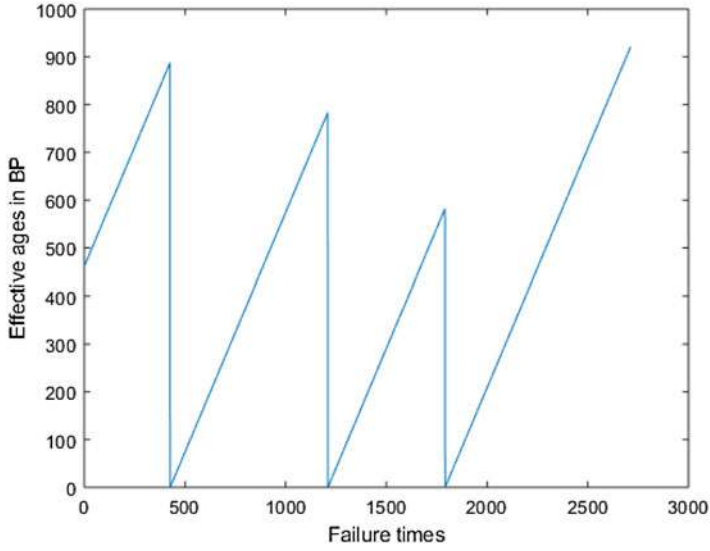


FIGURE 13. Virtual ages in the approximating BP model.

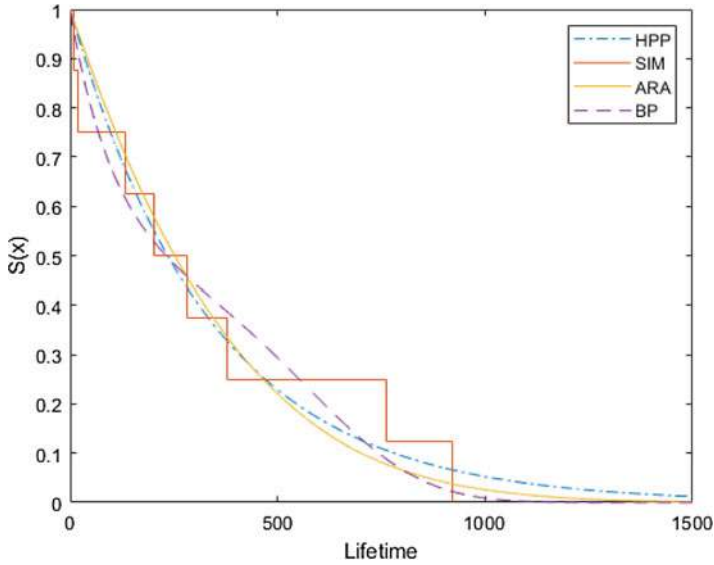


FIGURE 14. Survival functions of the inter-failure times of asset 012110.

The model parameters and reliability indicators are gathered in Table 6, and the unconditional survival functions are plotted in Figure 15. It can be observed that the exponential curve fits well the Kaplan–Meier estimate, signifying a weak dependence between intervals. The parameter of IAT_1 indicates a weak positive dependence, whereas ARA_∞ and BP suggest an increasing failure rate.

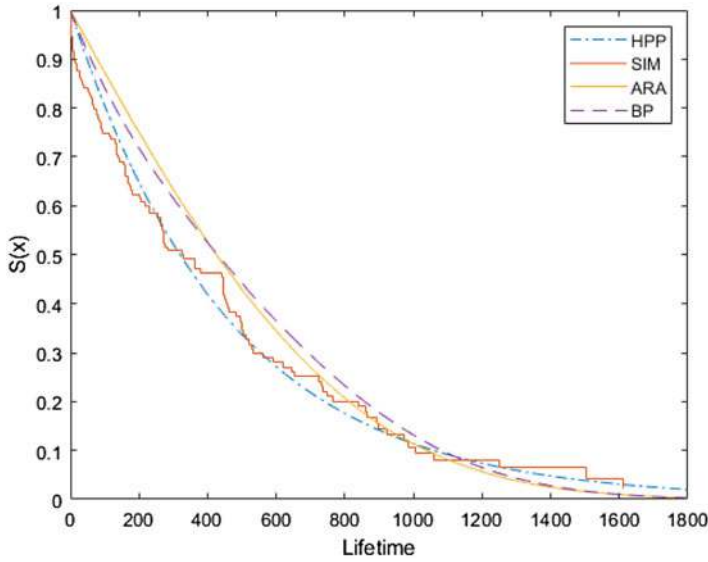


FIGURE 15. Survival function of the inter-failure times derived from the 16 most frequently failed assets.

TABLE 6. Reliability indicators for the 16 most frequently failed assets

	Parameters	A_0	$\mathbb{E}[X]$
ARA $_{\infty}$	$\alpha = 2.16 \times 10^{-12}$ $\beta = 3.5363$ $\rho = 0.2277$	1706.1	502.97
BP	$\alpha = 2.55 \times 10^{-7}$ $\beta = 2.2417$ $p = 0.4698$	575.07	509.63
HPP	460.27	0	460.27
IAT $_1$	$\theta = 0.1365$	0	440.12
SIM		0	440.12

TABLE 7. Distribution of the number of recorded CM

No.. failure	0	1	2	3	4	5	6	7	8	9	10	11
No. asset	821	419	178	95	51	17	11	8	2	4	0	2

4.4. Case Study 3: All Dwarf Signals

1,608 assets are investigated, half of which has no CM record. Table 7 gathers the distribution of number of CM of these assets.

The data set is therefore highly censored. Ignoring the heterogeneity and assume that the failure histories share the same model parameters, the inference results are gathered in Table 8.

Since the Kaplan–Meier estimator is not defined beyond the largest observation which is right-censored, the expected lifetime in the SIM model is calculated by changing the largest observation to a “death.” The survival functions given by each model are shown in

TABLE 8. Parameters and reliability indicators for all 1,608 dwarf signals

	Parameters	A_0	$\mathbb{E}[X]$
ARA $_{\infty}$	$\alpha = 0.2619$ $\beta = 0.1948$ $\rho = 0.9723$	13,448	474,963 (1,300 years)
BP	$\alpha = 0.0043$ $\beta = 0.5625$ $p = 1.0000$	0	26195 (72 years)
HPP	6227.1	0	6227.1 (17 years)
SIM		0	2576.1 (7 years)

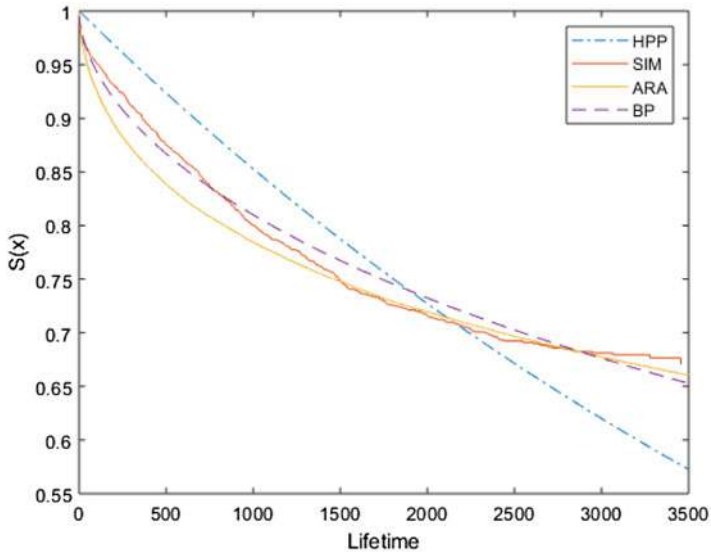


FIGURE 16. Survival functions derived from the failure times of all dwarf signals.

Figure 16. It can be observed that ARA $_{\infty}$ and BP differ in terms of the tail distribution (see Figure 17), resulting in a significant difference in the mean lifetime.

The fact that the empirical survival curve crosses its exponential fit once only, and from below, suggests strongly the possibility that the failure distribution has a decreasing failure rate (DFR) [26]. Both ARA $_{\infty}$ and BP exhibits the DFR via a Weibull shape parameter smaller than 1, and their unconditional survival function is much closer to the Kaplan–Meier estimate than the exponential fit. It has been pointed out in [6], however, that the observed DFR is possibly related to a heterogeneous population. In our case, the “observed” heterogeneity results directly from some measurable covariates, that is, working environment (average temperature, humidity and precipitation) and usage (total lighted time of the lamps), whereas the “unobserved” heterogeneity includes the variation in the quality or robustness of the lamps. For instance, some of the light bulbs are so robust that they can survive the fluctuation of voltage that could kill the others. Without external damage, their lifetime can be up to several decades.

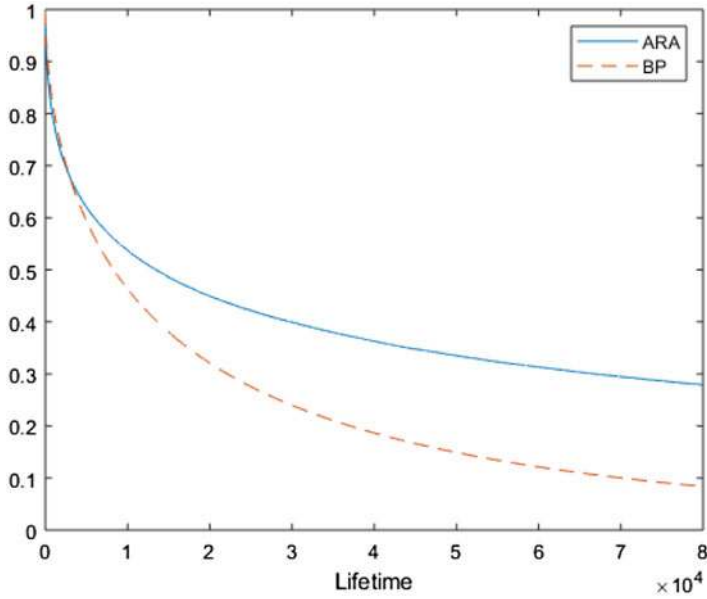


FIGURE 17. Tails of the survival function under ARA_∞ and BP models.

5. CONCLUSION

This paper introduces five approximating models of an SRP: two imperfect repair models (ARA_∞ and BP), two renewal models (SIM and HPP) and IAT_1 , constructed by the Kaplan–Meier estimated marginal distribution and a frank copula which captures the dependence structure between successive intervals.

The performances of these models are evaluated by investigating the amplitude of errors of mean interval length and correlations when the above-mentioned approximations are used. Further, their capabilities in the prognosis of RUL is examined using the level-set approach and the Gini index. It is hard to say which model is overall the best: the performances depend on the aging rate as well as the available data amount.

Then, three case studies on signal failure data provided by Bane NOR are conducted to complete the Monte Carlo simulations. It is observed that there may exist significant heterogeneity between the signals: a few assets exhibit the IFR, but when all assets are merged together, the DFR is revealed. The results may justify the current maintenance plan where no PM is implemented. Several limits of the case studies should be emphasized: first, the failure history of any individual signal is short; second, the “burn-in” phase of the lamps has been observed, but this can hardly be modeled by the proposed approximations; third, the DFR nature is related to the heterogeneity in the population, caused by different working conditions or distinctive frequency of usage.

Future research involves a thorough study of the heterogeneity, which plays an important role in survival analysis, especially the failure rate estimation. On the one hand, our Monte Carlo simulations suggest that the when heterogeneous virtual age processes (i.e., ARA_∞ processes that share same aging parameter β and repair effectiveness ρ but differ in the scale parameter α) are merged, traditional ML estimation will under-estimate the aging parameter and over-estimate the repair efficiency; on the other hand, the effect of numerous covariate (environmental conditions, frequency of usage and manufacture) could be analyzed through, for instance, the Cox proportional hazards model or random effect

survival models. These studies shall reinforce our understanding about the aging mechanism of signals and are helpful for the optimization of maintenance planning in signaling system.

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A. PROOF OF PROPOSITION 2.1

A.1 ABOUT RR₂ FUNCTION

A pair of real-valued random variables (X_1, X_2) and its density function $f(\cdot, \cdot)$ are called reverse rule of order 2 (RR₂) ([19]) if

$$f(x_1, y_1)f(x_2, y_2) \leq f(x_2, y_1)f(x_1, y_2), \tag{A.1}$$

whenever $x_1 > x_2$ and $y_1 > y_2$. According to Karlin and Rinott [17], this is equivalent to

$$f(x, y) \frac{\partial^2 f}{\partial x \partial y} \leq \frac{\partial f}{\partial x} \frac{\partial f}{\partial y}. \tag{A.2}$$

A.2 ARA_∞

The joint survival of two successive intervals in ARA in the stationary regime, conditioned on the previous virtual age satisfies Eq. (A.2). The proof is given below:

$$\begin{aligned} R(x, y) &= P(X_{n+1} \geq y, X_n \geq x | A_{n-1} = a) \\ &= \int_x^\infty P(X_{n+1} \geq y | X_n = u, A_{n-1} = a) f_{X_n | A_{n-1} = a}(u) du \\ &= \int_x^\infty P(X_{n+1} \geq y | A_n = (1 - \rho)(a + u)) \left(-\frac{d}{du} P(X_n \geq u | A_{n-1} = a) \right) du. \end{aligned}$$

Note $R_a(t) = P(X_n \geq t | A_{n-1} = a)$ and $R_u^b(t) = P(X_{n+1} \geq t | A_n = (1 - \rho)(a + u))$. The derivative of $R(x, y)$ with respect to x , y and mixed derivative are, respectively:

$$\begin{aligned}\frac{\partial R}{\partial x} &= -R_x^b(y) \left(-\frac{d}{dx} R_a(x) \right), \\ \frac{\partial R}{\partial y} &= \int_x^\infty \frac{d}{dy} R_u^b(y) \left(-\frac{d}{du} R_a(u) \right) du, \\ \frac{\partial^2 R}{\partial x \partial y} &= -\frac{d}{dy} R_x^b(y) \left(-\frac{d}{dx} R_a(x) \right).\end{aligned}$$

The derivative of survival function $R_x^b(t)$ is the product of failure rate and survival: $-(d/dy)R_x^b(y) = \lambda_x^b(y)R_x^b(y)$. Therefore, R is RR_2 if and only if

$$\int_x^\infty R_u^b(y) \left(-\frac{d}{du} R_a(u) \right) du \lambda_x^b(y) \leq \int_x^\infty \lambda_u^b(y) R_u^b(y) \left(-\frac{d}{du} R_a(u) \right) du.$$

Using the mean value theorem, the right side is reformulated as

$$\lambda_s^b(y) \int_x^\infty R_u^b(y) \left(-\frac{d}{du} R_a(u) \right) du, \quad s \in [x, \infty].$$

When the system is IFR, $\lambda_x^b(y) \leq \lambda_s^b(y)$, leading to A.2.

A.3 SRP

We introduce the following corollary when the SRP are identical:

COROLLARY A.1: *Note R , f the survival function and density function of an interval in the superposed RP. Define $\phi(t) = \int_t^\infty R(u)du$. Then, two successive intervals in the SRP in stationary regime is RR_2 -dependent if and only if*

$$\phi(t)f(t) \leq R(t)^2. \tag{A.3}$$

The proof is straightforward. Inequality (A.3) is easily satisfied in the case IFR since

$$\lambda(t) \int_t^\infty e^{-\Lambda(u)} du \leq \int_t^\infty \lambda(u) e^{-\Lambda(u)} du = R(t).$$