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# Sequential Detection of an Arbitrary Transient Change Profile by the FMA Test

Fatima Ezzahra Mana, Blaise Kevin Guepie, and Igor Nikiforov

Computer Science and Digital Society Laboratory Troyes, Université de Technologie de Troyes,  
Troyes, France

This article addresses the sequential detection of transient changes by using the finite moving average (FMA) test. It is assumed that a change occurs at an unknown (but nonrandom) change point and the duration of postchange period is finite and known. We relax the assumption that the profile of a transient change is chosen so that the log-likelihood ratios of the observations are associated random variables (r.v.s) in the prechange mode. Hence, the profile of the transient change is arbitrary and it is not necessarily of constant sign for a distribution with monotone likelihood ratio. A new upper bound for the worst-case probability of false alarm is proposed. It is shown that the optimization of the window-limited cumulative sum (CUSUM) test again leads to the FMA test. Three particular transient changes are considered: in the Gaussian mean, in the Gaussian variance, and in the parameter of exponential distribution. In the first case, a comparison between the bounds for the FMA test operating characteristics and the exact operating characteristics calculated by numerical integration is used to estimate the sharpness of the bounds. In the second and third cases, special attention is paid to the calculation of the FMA distribution in the case of arbitrary profile. The method of convolution is used to solve the problem.

## 1. INTRODUCTION

The criterion of the traditional sequential change detection is to minimize the (worst-case) mean detection delay for a given average run length (ARL) to false alarm. Such a criterion is well adapted to the quality control of the mass production process. The usage of the mean delay to detection or to false alarm is justified by the economic criterion of mass production process: some runs are short, some other runs are long, but after many repetitions, the optimum is reached. Unlike the traditional sequential change detection, which assumes that the postchange period is infinitely long, sometimes it is necessary to detect a change with an a priori upper-bounded detection delay. In such a scenario, all of the detections that exceed the required time to alert  $L$  are assumed missed.

For example, some safety/security critical systems such as drinking water distribution networks, air pollution monitoring systems, or satellite/inertial navigation systems should be equipped with special algorithmic tools, the main purpose of which is to detect an abrupt change of finite duration in the normal operation of the system (see Figure 1). Generally, these changes are called “transient.” In most cases, these transient changes result in a severe degradation of system safety/security when they are detected with the delay greater than the required time to alert  $L$ , defined by norms or standards. Let us consider the drinking water quality monitoring based on chlorine sensors; if we neglect the phenomenon of surface diffusion, the contaminated portion of water represents a cylinder inside the pipe. Hence, an efficient detection is possible before the cylinder passes the pipe section in which the chlorine sensor is installed.

Therefore, the criterion of the traditional sequential change detection—that is, the (worst-case) mean detection delay against the ARL to false alarm—cannot be used now and we have to switch to the probabilistic criterion of reliable detection; that is, the worst-case probability of missed detection against the worst-case probability of false alarm during a reference period (see Figure 1).

The sequential transient change detection (TCD) problem and its application have been considered in the literature (Zhigljavsky and Kraskovsky 1988; Broder and Schwartz 1990; Bakhache and Nikiforov 2000; Wang and Willett 2005b; Premkumar, Kumar, and Veeravalli 2010; Guépié, Fillatre, and Nikiforov 2012b, 2017; Pollak and Krieger 2013; Moustakides 2014; Do et al. 2017; Egea-Roca et al. 2018, 2022; Noonan and Zhigljavsky 2020, 2021; Tartakovsky et al. 2021; among others). The existing methods of sequential TCD can be divided into two groups with respect to the nature of the change point and/or the duration of transient change: Bayesian and non-Bayesian methods. In particular, it was shown in Bayesian and non-Bayesian setups that the Shewhart test minimizes the probability of missed detection among all tests with a given ARL to false alarm for the special case  $L = 1$  (Pollak and Krieger 2013; Moustakides 2014). The Shewhart test coincides with the finite moving average (FMA) test when  $L = 1$ . The case of geometrically distributed  $L \sim \text{Geom}(\psi)$  with a known parameter  $\psi$  and an optimal solution to the TCD problem in this case are considered in Tartakovsky et al. (2021). A non-Bayesian solution minimizing the worst-case probability of missed detection in a restricted class of truncated sequential probability ratio tests (SPRTs) provided that the

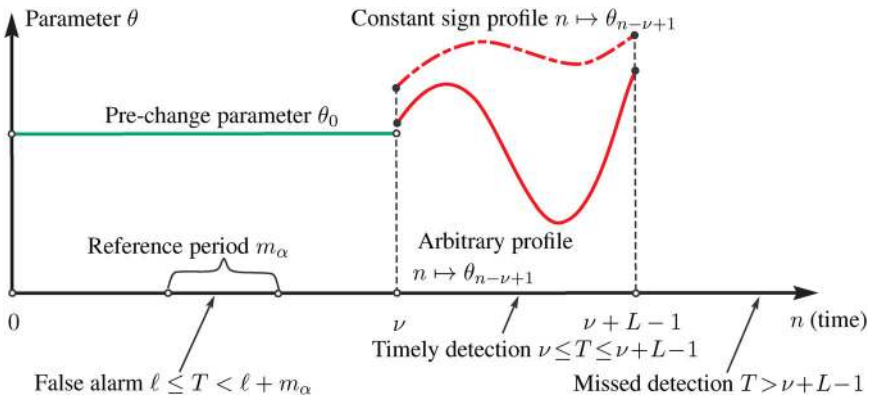


Figure 1. Constant sign and arbitrary profiles of transient change.

worst-case probability of false alarm during a reference period is upper bounded has been established in Guépié, Fillatre, and Nikiforov (2012b, 2017). It was shown that the window-limited (WL) cumulative sum (CUSUM) test with an optimal variable threshold is reduced to the FMA test. The idea to use several observations in a moving window of an inspection scheme as an alternative to the Shewhart test was initially proposed in Dudding and Jennett (1944). The FMA scheme with positive weights was introduced and intensively studied in Lai (1973, 1974) and Bohm and Hackl (1990) in the framework of the traditional quickest change detection for the ARL calculation by using the theory of associated r.v.s. The approximations of the probabilities of missed detection and false alarm of the FMA test in the Gaussian mean case (mainly with constant weighting coefficients) were studied in Noonan and Zhigljavsky (2020, 2021). A comparison of the FMA and fixed sample size tests as solutions to the TCD problem was given in Egea-Roca et al. (2022). Nevertheless, in the case of predefined duration  $L > 1$ , the (asymptotically) optimal solution is still open.

Let us consider the non-Bayesian framework, where the change point  $\nu$  is unknown but nonrandom. Let  $\{y_n\}_{n \geq 1}$  be a sequence of independent r.v.s and  $\nu$  be the index of the first postchange observation. The generative model of the transient change is given as follows:

$$y_n \sim \begin{cases} F_{\theta_0} & \text{if } 1 \leq n < \nu \\ F_{\theta_{n-\nu+1}} & \text{if } \nu \leq n \leq \nu + L - 1 \end{cases}, \quad (1.1)$$

where  $F_{\theta_0}$  is the prechange cumulative distribution function (c.d.f.),  $\theta_0$  is the prechange parameter,  $F_{\theta_1}, \dots, F_{\theta_L}$  are the known postchange c.d.f.s of the observations  $y_\nu, \dots, y_{\nu+L-1}$ , and the parametric set  $(\theta_1, \dots, \theta_L)$  is the profile of the transient change. It is worth noting that the generative model (1.1) is defined only for  $L$  observations after the change point because all that occurs after time  $\nu + L - 1$  is considered as a missed detection and is not of interest for this study. This generative model is illustrated by Figure 1.

In this article, we use the following criterion introduced in Guépié, Fillatre, and Nikiforov (2012b, 2017). Let us consider the stopping time  $T$ . It is assumed that  $\mathcal{P}_\nu$  (respectively  $\mathcal{P}_0$ ) is the joint distribution of the observations  $y_1, \dots, y_\nu, y_{\nu+1}, \dots$  when  $\nu < \infty$  (respectively  $\nu = \infty$ ). Let  $\mathbb{P}_\nu$  (respectively  $\mathbb{P}_0$ ) be the probability with respect to the distribution  $\mathcal{P}_\nu$  (respectively  $\mathcal{P}_0$ ). The worst-case probability of missed detection is

$$\bar{\mathbb{P}}_{\text{md}}(T) = \sup_{\nu \geq L} \mathbb{P}_\nu(T - \nu + 1 > L \mid T \geq \nu) \quad (1.2)$$

and the worst-case probability of false alarm during the reference period  $m_x$  is given as follows:

$$\bar{\mathbb{P}}_{\text{fa}}(T; m_x) = \sup_{\ell \geq L} \mathbb{P}_0(\ell \leq T < \ell + m_x). \quad (1.3)$$

This criterion is illustrated by Figure 1. In Guépié, Fillatre, and Nikiforov (2012b, 2017), the WL CUSUM test with a variable threshold was proposed as a solution to the TCD problem:

$$T_{\text{WL}} = \inf \left\{ n \geq L : \max_{1 \leq k \leq L} [Z_{n-k+1}^n - h_k] \geq 0 \right\}, \quad (1.4)$$

$$Z_{n-k+1}^n = \sum_{i=n-k+1}^n \ln \frac{f_{\theta_{k-n+i}}(y_i)}{f_{\theta_0}(y_i)}, \quad (1.5)$$

where  $h_k$  is a variable threshold,  $Z_{n-k+1}^n$  denotes the log-likelihood ratio (LLR) calculated for the time window  $[n-k+1, n]$ ,  $f_{\theta_0}$  (respectively  $f_{\theta_{k-n+i}}$ ) denotes the prechange probability density function (p.d.f.; respectively the postchange p.d.f.). It was shown in Guépié, Fillatre, and Nikiforov (2012b, 2017) that optimization of the variable threshold  $h_k$  with respect to the criteria (1.2) to (1.3) reduces the WL CUSUM test  $T_{\text{WL}}$  (1.4) to (1.5) to the FMA test under the assumption that the LLRs are associated r.v.s in the prechange mode. The FMA test stopping time is given as follows:

$$T_{\text{FMA}}(h) = \inf \{n \geq L : Z_{n-L+1}^n \geq h_L\}, \quad (1.6)$$

where  $h_L$  is a threshold. The article is organized as follows. In this section, we present a brief state of the art of the TCD problem and we state the problem that is treated in the article; that is, we define the criterion of optimality and the FMA test. In Section 2 we define the motivation and the contributions of the article. Section 3 is devoted to the new upper bound for  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}}; m_\alpha)$  calculated without the assumption that the LLRs are associated r.v.s in the prechange mode. It is also shown that the optimization of the WL CUSUM test with a variable threshold is valid for the new bound. In Section 4, the obtained theoretical results are applied to the Gaussian mean TCD problem. Here, the previous and new FMA test operating characteristics are compared against the exact operating characteristics calculated by numerical integration. Sections 5 and 6 are devoted to the TCD problem in the cases of Gaussian variance and exponential distribution. Here, the new FMA test operating characteristics are compared against the results of Monte Carlo simulations. The discussion of the obtained results is given in Section 7 and Section 8 concludes the article.

## 2. MOTIVATION AND CONTRIBUTION

The current article continues the study of the FMA test initiated in Guépié, Fillatre, and Nikiforov (2012b, 2012c, 2017). This work addresses new challenges relating to the general case of the profile  $(\theta_1, \dots, \theta_L)$  and to the application of findings to three different distributions. It follows from Guépié, Fillatre, and Nikiforov (2017, theorems 2 and 3) that the calculation of the worst-case probability of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}}; m_\alpha)$  and the optimization of the WL CUSUM test with a variable threshold (which leads to the FMA test) are heavily based on the concept of associated r.v.s (Lehmann 1966; Esary, Proschan, and Walkup 1967). Specifically, it is assumed that the profile is chosen so that the LLRs are associated r.v.s in the prechange mode (i.e., under the measure  $\mathcal{P}_0$ ). For the distributions with monotone likelihood ratio, this means that the profile  $(\theta_1, \dots, \theta_L)$  should be constant sign; that is,  $\theta_1 > \theta_0, \dots, \theta_L > \theta_0$ . Now we relax this assumption. Constant sign and arbitrary profiles of transient change are illustrated by Figure 1. The current article brings an upper bound for the worst-case FMA missed detection probability  $\bar{\mathbb{P}}_{\text{md}}(T)$  provided that the worst-case false alarm rate  $\bar{\mathbb{P}}_{\text{fa}}(T; m_\alpha)$  is upper bounded without limiting requirement that the LLRs are associated r.v.s in the prechange mode. However, the FMA test is an optimal solution only in a restricted class of truncated SPRT tests. When a lower bound of  $\bar{\mathbb{P}}_{\text{md}}(T)$  for all sequential tests will be

established, it will be possible to conclude whether the FMA operating characteristics are optimal and, if not, how far from optimality the FMA test is.

Therefore, the original contributions of the article are the following:

1. A new upper bound for the worst-case probability of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_\alpha)$  of the WL CUSUM test with a variable threshold is calculated without assumption that the LLRs are associated r.v.s in the prechange mode. Hence, the verification that the LLRs are associated r.v.s, which is not trivial for an arbitrary distribution, can be skipped. This new bound is applicable to an arbitrary profile  $(\theta_1, \dots, \theta_L)$ . It is shown that the previous and new bounds asymptotically converge when the threshold  $h_L \rightarrow \infty$ .
2. It is shown that the optimization of the WL CUSUM test with a variable threshold established in Guépié, Fillatre, and Nikiforov (2017, theorem 3) is also valid with this new bound. Therefore, the WL CUSUM test with a variable threshold is again reduced to the FMA test.
3. In the Gaussian mean case, the bounds for the FMA test operating characteristics are compared against the exact operating characteristics calculated by numerical integration of the multivariate Gaussian c.d.f. to show the sharpness of these bounds.
4. In the Gaussian variance case, the bounds for the FMA test operating characteristics are calculated by using the convolution of distributions corresponding to the positive and negative parts of the transient change profile. The operating characteristics are compared with the results of the Monte Carlo simulation.
5. Finally, in the case of exponential distribution, the bounds for the FMA test operating characteristics are reduced to those in the Gaussian variance case.

### 3. OPERATING CHARACTERISTICS FOR AN ARBITRARY DYNAMIC PROFILE

It follows from Guépié, Fillatre, and Nikiforov (2017, theorem 1) that the upper bound for the probability of missed detection  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$  of the WL CUSUM with a variable threshold and FMA tests is calculated without assumptions about the nature of LLRs and is given as

$$\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}}) \leq G(h_L) \stackrel{\text{def}}{=} \mathbb{P}_\nu(Z_\nu^{L+\nu-1} < h_L), \nu \geq L. \quad (3.1)$$

Because the assumption that the LLRs  $Z_L^L, \dots, Z_1^L, Z_{L+1}^{L+1}, \dots, Z_2^{L+1}, \dots, Z_{L+m_x-1}^{L+m_x-1}, \dots, Z_{m_x}^{L+m_x-1}$  are associated r.v.s in the prechange mode is relaxed now, we have to reestablish theorems 2 and 3 from Guépié, Fillatre, and Nikiforov (2017). The first assertion of this theorem is valid without the abovementioned assumption; that is, the worst-case probability of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_\alpha)$  for a given prechange period  $m_\alpha$  is given by

$$\bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_\alpha) \stackrel{\text{def}}{=} \sup_{\ell \geq L} \mathbb{P}_0(\ell \leq T_{\text{WL}} < \ell + m_\alpha) = \mathbb{P}_0(L \leq T_{\text{WL}} < L + m_\alpha). \quad (3.2)$$

The analogue of the second assertion is given by the following.

**Theorem 3.1.** *Let us consider the TCD problem (1.1) and the WL CUSUM test with a variable threshold (1.4) to (1.5). Then  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_\alpha)$  is upper bounded by*

$$\bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_\alpha) \leq \min\{1, m_\alpha p - (m_\alpha - L)^+ p^2\}, p = \mathbb{P}_0\left(\max_{1 \leq k \leq L} [Z_{L-k+1}^L - h_k] \geq 0\right), \quad (3.3)$$

where  $x^+ = \max\{0, x\}$ .

*Proof.* The proof of Theorem 3.1 is given in Appendix A.  $\square$

The analog of theorem 3 from Guépié, Fillatre, and Nikiforov (2017) is given by the following.

**Theorem 3.2.** *Let us consider the TCD problem (1.1) and the WL CUSUM test with a variable threshold (1.4) to (1.5). Suppose that the c.d.f.  $x \mapsto F_{Z,L}(x) = \mathbb{P}_0(Z_{n-L+1}^n < x)$  of the LLR  $Z_{n-L+1}^n$  under the measure  $\mathcal{P}_0$  and the c.d.f.  $x \mapsto G(x) = \mathbb{P}_\nu(Z_\nu^{\nu+L-1} < x)$  of the LLR  $Z_\nu^{\nu+L-1}$  under measure  $\mathcal{P}_\nu$  are continuous functions on  $]-\infty; \infty[$ . The optimal choice of the thresholds  $h_1, \dots, h_L$  is reduced to the following optimization problem:*

$$\begin{cases} \inf_{h_1, \dots, h_L} G(h_L) & = \bar{\alpha}_1 \\ \text{subject to } H(h_1, \dots, h_L) & = \bar{\alpha}_0 \end{cases}, \quad (3.4)$$

where

$$H(h_1, \dots, h_L) = \begin{cases} m_\alpha \tilde{p} & \text{if } 1 \leq m_\alpha \leq L \\ m_\alpha \tilde{p} - (m_\alpha - L) \tilde{p}^2 & \text{if } m_\alpha > L \geq 1 \text{ and } \tilde{p} \leq \frac{m_\alpha}{2(m_\alpha - L)}, \end{cases} \quad (3.5)$$

and  $\tilde{p} = \sum_{k=1}^{L-1} \mathbb{P}_0(Z_{L-k+1}^L \geq h_k) + 1 - F_{Z,L}(h_L)$ .

Then

1. The optimal solution  $\{h_i^*, i = 1, \dots, L\}$  of the minimization problem (3.4) to (3.5) is reached when  $h_1^* \rightarrow \infty, \dots, h_{L-1}^* \rightarrow \infty$  and

$$h_L^{\text{w.a.}} = h_L^* = \begin{cases} F_{Z,L}^{-1}\left(1 - \frac{\bar{\alpha}_0}{m_\alpha}\right) & \text{if } 1 \leq m_\alpha \leq L \\ F_{Z,L}^{-1}\left(1 - \frac{m_\alpha - \sqrt{m_\alpha^2 - 4(m_\alpha - L)\bar{\alpha}_0}}{2(m_\alpha - L)}\right) & \text{if } m_\alpha > L \geq 1 \text{ and,} \\ & \tilde{p} \leq \frac{m_\alpha}{2(m_\alpha - L)} \end{cases}, \quad (3.6)$$

where  $h_L^{\text{w.a.}}$  stands for the threshold calculated without assumption that the LLRs are associated r.v.s.

2. The smallest value  $\bar{\alpha}_1$  provided that the upper bound for the worst-case probability of false alarm is equal to a preassigned value  $\bar{\alpha}_0$  is given by

$$\bar{\alpha}_1 = G(h_L^{\text{w.a.}}(\bar{\alpha}_0)). \quad (3.7)$$

3. The stopping time  $T_{\text{WL}}$  of the optimal WL CUSUM test with a variable threshold is reduced to the stopping time  $T_{\text{FMA}}$  of the FMA test,

$$T_{\text{FMA}} = \inf\{n \geq L : Z_{n-L+1}^n \geq h_L^{\text{w.a.}}\}. \quad (3.8)$$

4. The upper bound for the probability of false alarm of the FMA test is given by

$$\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}}; m_\alpha) \leq \bar{\alpha}_0^{\text{w.a.}} = \min\left\{1, m_\alpha \tilde{p}_o - (m_\alpha - L)^+ \tilde{p}_o^2\right\}, \quad \tilde{p}_o = 1 - F_{Z,L}(h_L^{\text{w.a.}}). \quad (3.9)$$

*Proof.* The proof of Theorem 3.2 is given in Appendix B.  $\square$

**Remark 3.1.** The operating characteristic that completely characterizes the FMA test is the probability of missed detection  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$  as a function of the probability of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$ . Let us compare the asymptotic behavior of the upper bound  $\bar{\alpha}_1$  for  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$  as a function of  $\bar{\alpha}_0$  established in Theorem 3.2 without assumption that the LLRs are associated r.v.s in the prechange mode and the previous upper bound  $\bar{\alpha}_1$  for  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$  established in Guépié, Fillatre, and Nikiforov (2017, theorems 2 and 3) under such an assumption:

$$\bar{\alpha}_1 = G(h_L^{\text{a.}}) = G\left[F_{Z,L}^{-1}\left((1 - \bar{\alpha}_0)^{\frac{1}{m_\alpha}}\right)\right], \quad (3.10)$$

where  $h_L^{\text{a.}}$  stands for the threshold calculated with assumption that the LLRs are associated r.v.s. It follows immediately from the Taylor expansion that

$$\begin{aligned} G\left[F_{Z,L}^{-1}\left((1 - \bar{\alpha}_0)^{\frac{1}{m_\alpha}}\right)\right] &\sim G\left[F_{Z,L}^{-1}\left(1 - \frac{\bar{\alpha}_0}{m_\alpha}\right)\right] \\ &\sim G\left[F_{Z,L}^{-1}\left(1 - \frac{m_\alpha - \sqrt{m_\alpha^2 - 4(m_\alpha - L)\bar{\alpha}_0}}{2(m_\alpha - L)}\right)\right] \end{aligned} \quad (3.11)$$

as  $\bar{\alpha}_0 \rightarrow 0$ . Therefore, the operating characteristics of the FMA test with and without the assumption that the LLRs are associated r.v.s in the prechange mode are asymptotically equal.

The nonasymptotic numerical comparison will be discussed in the next section.

#### 4. TRANSIENT CHANGE IN THE GAUSSIAN MEAN

Let us first rewrite the generative TCD model (1.1) for the Gaussian mean case

$$y_n \sim \begin{cases} \mathcal{N}(\theta_0, \sigma^2) & \text{if } 1 \leq n < \nu \\ \mathcal{N}(\theta_{n-\nu+1}, \sigma^2) & \text{if } \nu \leq n \leq \nu + L - 1 \end{cases}, \quad (4.1)$$

where  $\mathcal{N}(\mu, \sigma^2)$  stands for the Gaussian law with mean  $\mu$  and variance  $\sigma^2$  and the mean vector  $(\theta_1, \dots, \theta_L)$  stands for the transient change profile. It follows from (3.8) that the FMA stopping time is defined as

$$T_{\text{FMA}} = \inf\left\{n \geq L : S_n = \sum_{i=n-L+1}^n \theta_{L-n+i} y_i \geq h\right\}. \quad (4.2)$$

The upper bound  $\bar{\alpha}_0^{\text{a.}}$  for  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$  calculated in Guépié, Fillatre, and Nikiforov (2017) under the assumption that the LLRs are associated r.v.s in the prechange mode is given by



$$\bar{\alpha}_0^a = 1 - \left[ \Phi \left( \frac{h}{\sigma \|\theta^a\|_2} \right) \right]^{m_x}, \quad (4.3)$$

where  $\|\theta^a\|_2$  is the Euclidean norm of the vector  $\theta^a$  with constant sign coefficients:  $\theta_1 > \theta_0, \dots, \theta_L > \theta_0$ . It follows immediately from Theorem 3.1 that the upper bound  $\bar{\alpha}_0^{\text{w.a.}}$  (3.9) without the assumption that the LLRs are associated r.v.s is given by

$$\bar{\alpha}_0^{\text{w.a.}} = \min \left\{ 1, m_x \tilde{p}_o - (m_x - L)^+ \tilde{p}_o^2 \right\}, \tilde{p}_o = 1 - \Phi \left( \frac{h}{\sigma \|\theta^{\text{w.a.}}\|_2} \right), \quad (4.4)$$

where  $\theta^{\text{w.a.}}$  is a vector with arbitrary coefficients.

It follows from (4.3) and (4.4) that both upper bounds depend only on the Euclidean norm of the vector  $\theta$  of transient change profile. Let us compare the upper bounds  $\bar{\alpha}_0^a$  and  $\bar{\alpha}_0^{\text{w.a.}}$  for the probability of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$  calculated with and without the assumption that the LLRs are associated with the true probability given as follows:

$$\begin{aligned} \bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}}) &= \mathbb{P}_0(L \leq T_{\text{FMA}} < L + m_x) = 1 - \mathbb{P}_0 \left( \bigcap_{n=L}^{L+m_x-1} \{S_n < h\} \right) \\ &= 1 - F(h, \dots, h), \end{aligned} \quad (4.5)$$

where  $(x_1, \dots, x_{m_x}) \rightarrow F(x_1, \dots, x_{m_x})$  is the joint c.d.f. of the random vector  $(S_L, \dots, S_{L+m_x-1})$  of size  $m_x$ , which obeys the Gaussian distribution  $\mathcal{N}(0, V)$ , calculated by using a numerical procedure from Matlab (MATLAB 2019). To define the Toeplitz covariance matrix  $V$  of size  $(m_x \times m_x)$ , it is necessary to compute the autocovariance function  $R(\ell)$  of the random sequence  $\{S_n\}_{n \geq L}$ :

$$R(\ell) = \sigma^2 \sum_{j=1}^{L-\ell} \theta_j \theta_{j+\ell}, \quad \ell = 0, 1, \dots, L-1, \quad (4.6)$$

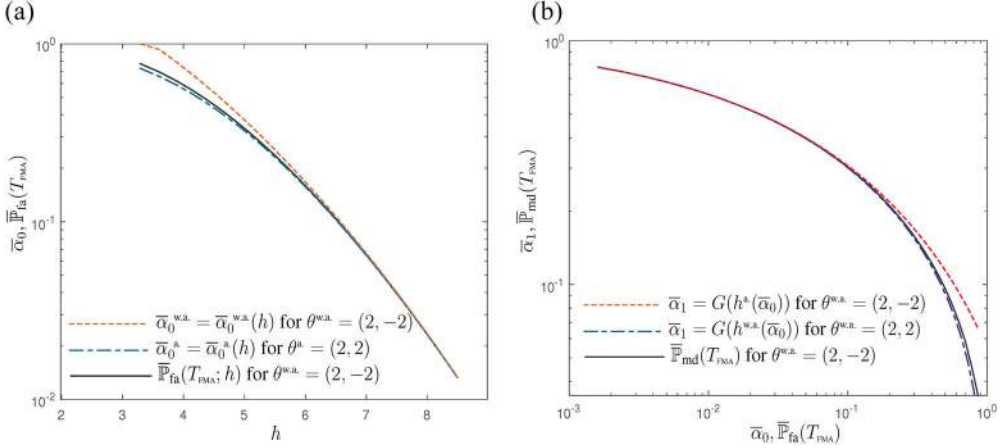
and, next, to fill the missing left-to-right descending diagonals with zeros, if  $m_x > L$ .

To illustrate the sharpness of the upper bounds  $\bar{\alpha}_0^{\text{w.a.}}$  and  $\bar{\alpha}_1$ , let us consider the following example of the TCD problem (4.1):  $L=2$ ,  $\theta^a = (2, 2)$ ,  $\theta^{\text{w.a.}} = (2, -2)$ ,  $\sigma^2 = 1$ ,  $m_x = 10$ . The results of comparison are presented in Figure 2 (a). Here, the upper bounds  $\bar{\alpha}_0^a$  and  $\bar{\alpha}_0^{\text{w.a.}}$  are compared against the numerically calculated probability of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$  given by (4.5). First, it is confirmed that the upper bound  $\bar{\alpha}_0^a$  calculated for the vector  $\theta^a = (2, 2)$  is not applicable to the vector  $\theta^{\text{w.a.}} = (2, -2)$ , though  $\|\theta^a\|_2 = \|\theta^{\text{w.a.}}\|_2 = 2\sqrt{2}$ . Second, both bounds converge to the true probability of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$  rather quickly.

Let us continue this example and compare the upper bounds for the operating characteristics  $\bar{\alpha}_1 = G(h^{\text{w.a.}}(\bar{\alpha}_0))$  and  $\bar{\alpha}_1 = G(h^a(\bar{\alpha}_0))$  with the numerically calculated probability of missed detection  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$  as a function of  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$ . The probability of missed detection is defined as follows:

$$\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}}) = \sup_{\nu \geq L} \mathbb{P}_\nu(T_{\text{FMA}} - \nu + 1 > L | T_{\text{FMA}} \geq \nu) = \sup_{\nu \geq L} \frac{\mathbb{P}_\nu \left( \bigcap_{n=L}^{L+\nu-1} \{S_n < h\} \right)}{\mathbb{P}_\nu \left( \bigcap_{n=L}^{\nu-1} \{S_n < h\} \right)}, \quad (4.7)$$

where  $\mathbb{P}_\nu(\bigcap_{n=L}^{\nu-1} \{S_n < h\}) = \mathbb{P}_0(\bigcap_{n=L}^{\nu-1} \{S_n < h\})$ , the random vector  $(S_L, \dots, S_{\nu-1})$  obeys the multivariate zero-mean Gaussian distribution  $\mathcal{N}(0, V_1)$ , and the random vector



**Figure 2.** (a) The upper bounds  $\bar{\alpha}_0^a$  and  $\bar{\alpha}_0^{w.a.}$  (with and without the assumption that the LLRs are associated) and the true value of  $\mathbb{P}_{fa}(T_{FMA})$  as functions of the threshold  $h$ . (b) The upper bounds  $\bar{\alpha}_1 = G(h^a(\bar{\alpha}_0))$  and  $\bar{\alpha}_1 = G(h^{w.a.}(\bar{\alpha}_0))$  (with and without the assumption that the LLRs are associated) and the true value of  $\mathbb{P}_{md}(T_{FMA})$  as functions of  $\bar{\alpha}_0 = \mathbb{P}_{fa}(T_{FMA})$ .

$(S_L, \dots, S_{L+\nu-1})$  obeys the multivariate Gaussian distribution  $\mathcal{N}(\eta, V_2)$ . The Toeplitz covariance matrices  $V_1$  and  $V_2$  of sizes  $((\nu - L) \times (\nu - L))$  and  $(\nu \times \nu)$  are calculated following (4.6). The mean vector  $\eta$  is composed of two parts. The first  $\nu - L$  coefficients are equal to zero,  $\eta_1 = \eta_2 = \dots = \eta_{\nu-L} = 0$ , and the last  $L$  coefficients are calculated as follows:

$$\eta_{k+\nu-L} = \sum_{m=1}^k \theta_{k-m+1} \theta_{L-m+1}, \quad k = 1, \dots, L. \quad (4.8)$$

Finally, the maximum  $\max_{L \leq \nu \leq \nu_0}$  is used instead of  $\sup_{\nu \geq L}$  in (4.7), where  $\nu_0$  is defined empirically when convergence is reached. The results of this comparison are presented in Figure 2 (b). It follows from Figure 2 (b) that, beginning from  $\bar{\alpha}_0 = 10^{-1}$ , both bounds are practically equal.

## 5. TRANSIENT CHANGE IN THE GAUSSIAN VARIANCE

Let us rewrite the generative TCD model (1.1) for the Gaussian variance case

$$y_n \sim \begin{cases} \mathcal{N}(0, \sigma_0^2) & \text{if } 1 \leq n < \nu \\ \mathcal{N}(0, \sigma_{n-\nu+1}^2) & \text{if } \nu \leq n \leq \nu + L - 1 \end{cases}, \quad (5.1)$$

where  $(\sigma_1^2, \dots, \sigma_L^2)$  stands for the transient change profile. It follows immediately from (3.8) that the FMA stopping time is defined as

$$T_{FMA} = \inf \left\{ n \geq L : S_n = \sum_{i=n-L+1}^n \left( \frac{1}{\sigma_0^2} - \frac{1}{\sigma_{L-n+i}^2} \right) y_i^2 \geq h \right\}. \quad (5.2)$$

It follows from (3.1) that

$$\bar{\mathbb{P}}_{md}(T_{FMA}) \leq \bar{\alpha}_1 = G(h) = \mathbb{P}_\nu(S_{L+\nu-1} < h) = \mathbb{P}_\nu \left[ \sum_{i=\nu}^{L+\nu-1} \left( \frac{1}{\sigma_0^2} - \frac{1}{\sigma_{i-\nu+1}^2} \right) y_i^2 < h \right]. \quad (5.3)$$

Because the r.v.s  $y_\nu, \dots, y_{L+\nu-1}$  obey the Gaussian distributions  $\mathcal{N}(0, \sigma_1^2), \dots, \mathcal{N}(0, \sigma_L^2)$ , equation (5.3) can be rewritten as follows:

$$\bar{\alpha}_1 = G(h) = \mathbb{P} \left( \sum_{i=1}^L d_i^1 \xi_i^2 < h \right), \quad (5.4)$$

where  $d_i^1 = \frac{\sigma_i^2}{\sigma_0^2} - 1$ ,  $\xi_1, \dots, \xi_L$  are independent standard Gaussian r.v.s,  $\xi_i \sim \mathcal{N}(0, 1)$ , and  $\xi_i^2 \sim \chi_1^2$  obeys the chi-square distribution with 1 degree of freedom.

It follows from (3.9) that the upper bound for  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}}; m_\alpha)$  is given by

$$\bar{\alpha}_0^{\text{w.a.}} = \min \left\{ 1, m_\alpha \tilde{p}_o - (m_\alpha - L) \tilde{p}_o^2 \right\}, \quad \tilde{p}_o = 1 - \mathbb{P}_0 \left[ \sum_{i=1}^L \left( \frac{1}{\sigma_0^2} - \frac{1}{\sigma_i^2} \right) y_i^2 < h \right]. \quad (5.5)$$

Because the r.v.s  $y_1, \dots, y_L$  obey the Gaussian distribution  $\mathcal{N}(0, \sigma_0^2)$ , the term  $\tilde{p}_o$  from (5.5) can be rewritten as follows:

$$\tilde{p}_o = 1 - F_{Z,L}(h) = 1 - \mathbb{P} \left( \sum_{i=1}^L d_i^0 \xi_i^2 < h \right), \quad (5.6)$$

where  $d_i^0 = 1 - \frac{\sigma_0^2}{\sigma_i^2}$ ,  $\xi_1, \dots, \xi_L$  are independent standard Gaussian r.v.s,  $\xi_i \sim \mathcal{N}(0, 1)$ , and  $\xi_i^2 \sim \chi_1^2$  obeys the chi-square distribution with 1 degree of freedom.

To compute the upper bounds  $\bar{\alpha}_0^{\text{w.a.}}$  and  $\bar{\alpha}_1$  for  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}}, m_\alpha)$  and  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$ , it is necessary to find the c.d.f.s  $x \mapsto G(x)$  and  $x \mapsto F_{Z,L}(x)$ . The exact c.d.f. of the quadratic form in Gaussian variables  $\sum_{i=1}^L d_i \xi_i^2$  with arbitrary weighting coefficients is available only for some special cases, and its calculation in the general case is very difficult (see chapter 4.6 in Mathai and Provost 1992). For this reason, the following procedure is proposed.

First, let us rearrange the elements of the vector  $(\sigma_1^2, \dots, \sigma_L^2)$  and divide it into two subvectors  $(\sigma_{(1)}^{2+}, \dots, \sigma_{(L_1)}^{2+})$  and  $(\sigma_{(1)}^{2-}, \dots, \sigma_{(L_2)}^{2-})$ , where  $L_1 + L_2 = L$ . The first subvector represents the “positive” part of the profile—that is,  $\sigma_{(1)}^{2+} > \sigma_0^2, \dots, \sigma_{(L_1)}^{2+} > \sigma_0^2$ —and the second subvector represents the “negative” part of the profile; that is,  $\sigma_{(1)}^{2-} < \sigma_0^2, \dots, \sigma_{(L_2)}^{2-} < \sigma_0^2$ . Let us define four r.v.s  $\Xi^{1,+} = \sum_{i=1}^{L_1} d_{(i)}^{1,+} \xi_i^2$  (respectively  $\Xi^{0,+} = \sum_{i=1}^{L_1} d_{(i)}^{0,+} \xi_i^2$ ) and  $\Xi^{1,-} = \sum_{i=1}^{L_2} -d_{(i)}^{1,-} \xi_i^2$  (respectively  $\Xi^{0,-} = \sum_{i=1}^{L_2} -d_{(i)}^{0,-} \xi_i^2$ ) to calculate  $x \mapsto G(x)$  (respectively  $x \mapsto F_{Z,L}(x)$ ). The coefficients  $d_{(i)}^{0,+}$  and  $d_{(i)}^{1,+}$  are calculated for the subvector  $(\sigma_{(1)}^{2+}, \dots, \sigma_{(L_1)}^{2+})$ . The coefficients  $d_{(i)}^{0,-}$  and  $d_{(i)}^{1,-}$  are calculated for the subvector  $(\sigma_{(1)}^{2-}, \dots, \sigma_{(L_2)}^{2-})$ , and  $\xi_i, \varsigma_i \sim \mathcal{N}(0, 1)$  are independent standard Gaussian r.v.s.

Hence, the abovementioned c.d.f.s can be rewritten as follows:

$$\begin{aligned} G(h) &= \mathbb{P} \left( \sum_{i=1}^L d_i^1 \xi_i^2 < h \right) = \mathbb{P} \left( \sum_{i=1}^{L_1} d_{(i)}^{1,+} \xi_i^2 - \sum_{i=1}^{L_2} -d_{(i)}^{1,-} \xi_i^2 \leq h \right) \\ &= \mathbb{P}(\Xi^{1,+} - \Xi^{1,-} \leq h) \end{aligned} \quad (5.7)$$

$$\begin{aligned} F_{Z,L}(h) &= \mathbb{P} \left( \sum_{i=1}^L d_i^0 \xi_i^2 < h \right) = \mathbb{P} \left( \sum_{i=1}^{L_1} d_{(i)}^{0,+} \xi_i^2 - \sum_{i=1}^{L_2} -d_{(i)}^{0,-} \xi_i^2 \leq h \right) \\ &= \mathbb{P}(\Xi^{0,+} - \Xi^{0,-} \leq h). \end{aligned} \quad (5.8)$$

Next, the use of a rather precise approximation for the c.d.f.s of the r.v.s  $\Xi^{0,+}$ ,  $\Xi^{1,+}$ ,  $\Xi^{0,-}$ , and  $\Xi^{1,-}$  is proposed assuming that all of the weighting coefficients are positive; that is,  $d_{(i)}^{0,+} > 0$ ,  $d_{(i)}^{1,+} > 0$ ,  $-d_{(i)}^{0,-} > 0$  and  $-d_{(i)}^{1,-} > 0$  (Gabler and Wolff 1987; Mathai and Provost 1992):

$$F_{\Xi}(x) = \mathbb{P} \left( \sum_{i=1}^n \tilde{d}_{(i)} \xi_i^2 < x \right) = \min\{H_1(x), H_2(x)\}, \quad (5.9)$$

where  $n = L_1$  or  $n = L_2$ ,  $\xi_i \sim \mathcal{N}(0, 1)$ ,  $\tilde{d}_i = d_i/d > 0$  stands for the normalized coefficients calculated for  $d_i = \pm d_{(i)}^{0,\pm}$  or for  $d_i = \pm d_{(i)}^{1,\pm}$ ,  $\sum_{i=1}^n \tilde{d}_i = 1$ , and  $d = \sum_{i=1}^n d_i$ ,

$$H_1(x) = \sum_{i=1}^n \tilde{d}_i \frac{\gamma(1/(2\tilde{d}_i), x/(2\tilde{d}_i))}{\Gamma(1/(2\tilde{d}_i))}, \quad (5.10)$$

$\Gamma(x)$  stands for the gamma function,  $\gamma(p, x) = \int_0^x u^{p-1} e^{-u} du$  stands for the incomplete gamma function, and

$$H_2(x) = F_{\chi^2}(x/\delta_d; n), \quad \delta_d = \left( \prod_{i=1}^n \tilde{d}_i \right)^{1/n}, \quad (5.11)$$

where  $F_{\chi^2}(x; n)$  stands for the c.d.f. of the chi-square distribution with  $n$  degrees of freedom.

Finally, the convolution product is used to calculate the c.d.f.s  $x \mapsto G(x)$  and  $x \mapsto F_{Z,L}(x)$ . Therefore, we get

$$G(x) = \mathbb{P}(\Xi^{1,+} - \Xi^{1,-} \leq x) = \int_0^{\infty} F_{\Xi^{1,+}}(t+x) f_{\Xi^{1,-}}(t) dt, \quad (5.12)$$

and

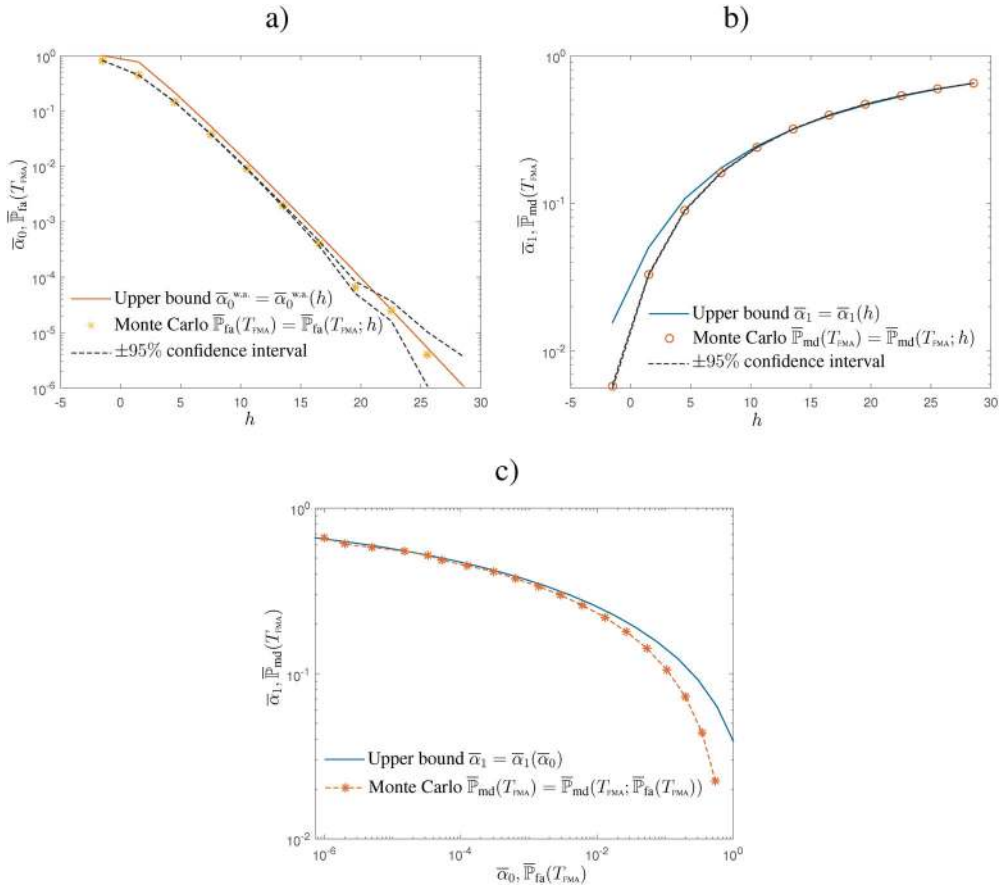
$$F_{Z,L}(x) = \mathbb{P}(\Xi^{0,+} - \Xi^{0,-} \leq x) = \int_0^{\infty} F_{\Xi^{0,+}}(t+x) f_{\Xi^{0,-}}(t) dt, \quad (5.13)$$

where  $x \mapsto F_{\Xi^{1,+}}(x)$  (respectively  $x \mapsto F_{\Xi^{0,+}}(x)$ ) denotes the c.d.f. of  $\Xi^{1,+}$  (respectively  $\Xi^{0,+}$ ) and  $x \mapsto f_{\Xi^{1,-}}(t)$  (respectively  $x \mapsto f_{\Xi^{0,-}}(t)$ ) denotes the p.d.f. of  $\Xi^{1,-}$  (respectively  $\Xi^{0,-}$ ). The p.d.f.s  $t \mapsto f_{\Xi^{1,-}}(t) = \frac{dF_{\Xi^{1,-}}(t)}{dt}$  and  $t \mapsto f_{\Xi^{0,-}}(t) = \frac{dF_{\Xi^{0,-}}(t)}{dt}$  for equations (5.12) and (5.13) are calculated by the numerical differentiation of the c.d.f.s (5.9).

The bottleneck point of the proposed numerical scheme is the application of the convolution products (5.12) to (5.13) to the calculation of the c.d.f.s  $x \mapsto G(x)$  and  $x \mapsto F_{Z,L}(x)$ . The potential computational burden of these c.d.f.s' approximations are due to the numerical integration and differentiation algorithms. It is necessary to find an appropriate upper limit  $c$  of the integrals in (5.12) and (5.13) instead of  $\infty$ . On the one hand, the remainder terms  $\int_c^{\infty} F_{\Xi^{1,+}}(t+x) f_{\Xi^{1,-}}(t) dt$ ,  $\int_c^{\infty} F_{\Xi^{0,+}}(t+x) f_{\Xi^{0,-}}(t) dt$  must be negligible but, on the other hand, a large upper limit  $c$  produces a considerable volume of calculation. Let us consider the following test example: if  $\xi_1, \xi_2 \sim \chi_2^2$ , then  $\xi_1 - \xi_2 \sim \text{Laplace}(0, 2)$ . The comparison between the c.d.f. obtained by the numerical scheme with  $c=100$  and the Laplace c.d.f. shows that the upper bounds for

$\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}}; m_x)$  and  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$  can be reliably calculated in the interval  $[10^{-12}, 1]$  with a relative error induced by the numerical scheme of order  $10^{-9}$ .

Let us consider the following example of the TCD problem (5.1):  $L = 10$ ,  $\sigma_0^2 = 1$ ,  $(\sigma_1^2, \dots, \sigma_L^2) = (10, 8, 6, 2, 8, 0.1, 0.2, 0.4, 0.6, 0.1)$ ,  $m_x = 20$ . For this TCD problem, the numerical calculation of the probabilities of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$  and missed detection  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$  is difficult due to the absence of practicable methods. For this reason, the probabilities  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$  and  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$  are estimated by using a  $10^6$ -repetition Monte Carlo simulation. To illustrate the sharpness of the upper bounds  $\bar{\alpha}_0^{\text{w.a.}}$  and  $\bar{\alpha}_1$ , their comparison with the probabilities  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$  and  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$  estimated by simulation is shown in Figure 3. It follows that the bounds  $\bar{\alpha}_0^{\text{w.a.}}$  and  $\bar{\alpha}_1$  converge rather quickly to the probabilities of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$  and missed detection  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$ .



**Figure 3.** (a) The upper bound  $\bar{\alpha}_0^{\text{w.a.}}$  (without the assumption that the LLRs are associated) and the probability  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$  obtained by Monte Carlo simulation as functions of the threshold  $h$ . (b) The upper bound  $\bar{\alpha}_1$  and the probability  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$  obtained by Monte Carlo simulation as functions of the threshold  $h$ . (c) The upper bound  $\bar{\alpha}_1$  and the probability  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$  obtained by Monte Carlo simulation as functions of  $\bar{\alpha}_0$  and  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$ .

## 6. TRANSIENT CHANGE IN THE PARAMETER OF EXPONENTIAL DISTRIBUTION

Let us rewrite the generative TCD model (1.1) for the exponential distribution  $\text{Exp}(\lambda)$  :

$$y_n \sim \begin{cases} \text{Exp}(\lambda_0) & \text{if } 1 \leq n < \nu \\ \text{Exp}(\lambda_{n-\nu+1}) & \text{if } \nu \leq n \leq \nu + L - 1 \end{cases}, \quad (6.1)$$

where  $(\lambda_1, \dots, \lambda_L)$  stands for the transient change profile. The p.d.f. of  $\text{Exp}(\lambda)$  is given by

$$f(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}. \quad (6.2)$$

It follows immediately from (3.8) that the FMA stopping time is defined as

$$T_{\text{FMA}} = \inf \left\{ n \geq L : S_n = \sum_{i=n-L+1}^n (\lambda_0 - \lambda_{L-n+i}) y_i \geq h \right\}. \quad (6.3)$$

As in Section 5, to compute the upper bounds  $\bar{\alpha}_0^{\text{w.a.}}$  and  $\bar{\alpha}_1$  for  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}}; m_x)$  and  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$ , it is necessary to find the c.d.f.s  $x \mapsto G(x)$  and  $x \mapsto F_{Z,L}(x)$ . The distribution of the sum of independent exponential r.v.s  $\sum_{i=1}^n y_i$ , where  $y_i \sim \text{Exp}(\lambda_i)$ , obeys the hypoexponential distribution (or the generalized Erlang distribution; Ross 2011). The exact c.d.f. of the hypoexponential distribution is available only for some special cases of  $\lambda_1, \dots, \lambda_n$ , and its calculation in the general case is very difficult. It follows from the definition of the chi-square distribution with 2 degrees of freedom that  $\xi \sim \text{Exp}(1/2) \sim \chi_2^2$  and, hence,  $\xi \sim \text{Exp}(\lambda) \sim (1/2\lambda)\chi_2^2$ . Therefore, we again use the approximation proposed in Gabler and Wolff (1987) and Mathai and Provost (1992).

It follows from (3.1) that

$$\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}}) \leq \bar{\alpha}_1 = G(h) = \mathbb{P}_\nu(S_{L+\nu-1} < h) = \mathbb{P}_\nu \left[ \sum_{i=\nu}^{L+\nu-1} (\lambda_0 - \lambda_{i-\nu+1}) y_i < h \right]. \quad (6.4)$$

Taking into account the previous paragraph, (6.4) can be rewritten as follows:

$$\bar{\alpha}_1 = G(h) = \mathbb{P} \left( \sum_{i=1}^L d_i^1 (\xi_i^2 + \varsigma_i^2) < h \right), \quad (6.5)$$

where  $d_i^1 = \frac{1}{2} \left( \frac{\lambda_0}{\lambda_i} - 1 \right)$ ,  $\xi_1, \dots, \xi_L, \varsigma_1, \dots, \varsigma_L$  are independent standard Gaussian r.v.s,  $\xi_i, \varsigma_i \sim \mathcal{N}(0, 1)$ , and  $\xi_i^2 + \varsigma_i^2 \sim \chi_2^2$  obeys the chi-square distribution with 2 degrees of freedom.

It follows from (3.9) that the upper bound for  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}}; m_x)$  is given by

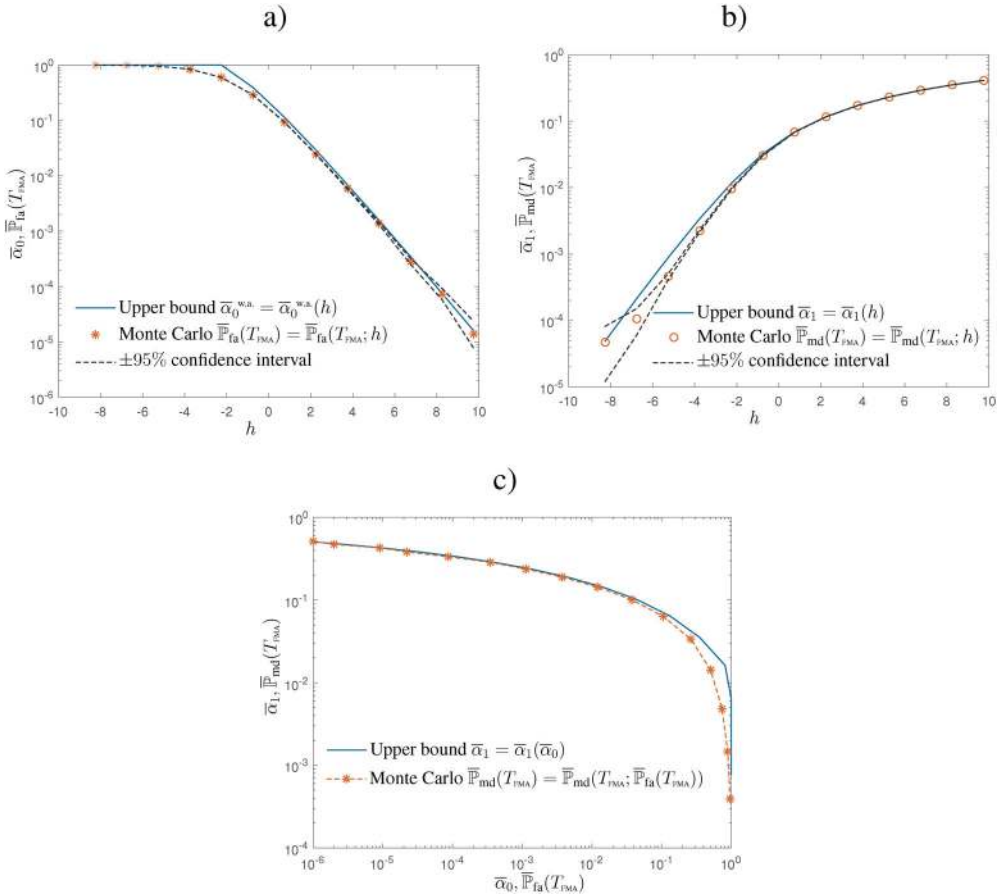
$$\bar{\alpha}_0^{\text{w.a.}} = \min \left\{ 1, m_x \tilde{p}_o - (m_x - L)^+ \tilde{p}_o^2 \right\}, \quad \tilde{p}_o = 1 - \mathbb{P}_0 \left[ \sum_{i=1}^L (\lambda_0 - \lambda_i) y_i < h \right]. \quad (6.6)$$

By the same reason as given previously, the term  $\tilde{p}_o$  from (6.6) can be rewritten as follows:

$$\tilde{p}_o = 1 - F_{Z,L}(h) = 1 - \mathbb{P} \left( \sum_{i=1}^L d_i^0 (\xi_i^2 + \varsigma_i^2) < h \right), \quad (6.7)$$

where  $d_i^0 = \frac{1}{2} \left(1 - \frac{\lambda_i}{\lambda_0}\right)$ ,  $\xi_1, \dots, \xi_L, \varsigma_1, \dots, \varsigma_L$  are independent standard Gaussian r.v.s,  $\xi_i, \varsigma_i \sim \mathcal{N}(0, 1)$ , and  $\xi_i^2 + \varsigma_i^2 \sim \chi_2^2$  obeys the chi-square distribution with 2 degrees of freedom.

As in Section 5, let us rearrange the elements of the vector  $(\lambda_1, \dots, \lambda_L)$  and divide it into two subvectors  $(\lambda_{(1)}^+, \dots, \lambda_{(L_1)}^+)$  and  $(\lambda_{(1)}^-, \dots, \lambda_{(L_2)}^-)$ , where  $L_1 + L_2 = L$ . The first subvector represents the “positive” part of the profile; that is, the part of positive weighting coefficients  $(\lambda_0 - \lambda_i) > 0$  in the sums (6.4) and (6.6) or  $\lambda_{(1)}^+ < \lambda_0, \dots, \lambda_{(L_1)}^+ < \lambda_0$ . The second subvector represents the “negative” part of the profile; that is,  $\lambda_{(1)}^- > \lambda_0, \dots, \lambda_{(L_2)}^- > \lambda_0$ . Next, equations (5.7) to (5.13) are used again with a suitable adaptation of the coefficients  $d_{(i)}^{0,+}$  and  $d_{(i)}^{1,+}$  for the subvector  $(\lambda_{(1)}^+, \dots, \lambda_{(L_1)}^+)$  and the coefficients  $d_{(i)}^{0,-}$  and  $d_{(i)}^{1,-}$  for the subvector  $(\lambda_{(1)}^-, \dots, \lambda_{(L_2)}^-)$  and by replacing  $L_1$  (respectively  $L_2$ ) with  $2L_1$  (respectively  $2L_2$ ).



**Figure 4.** (a) The upper bound  $\bar{\alpha}_0^{w.a.}$  (without the assumption that the LLRs are associated) and the probability  $\bar{\mathbb{P}}_{fa}(T_{FMA})$  obtained by a  $10^6$ -repetition Monte Carlo simulation as functions of the threshold  $h$ . (b) The upper bound  $\bar{\alpha}_1$  and the probability  $\bar{\mathbb{P}}_{md}(T_{FMA})$  obtained by a  $10^6$ -repetition Monte Carlo simulation as functions of the threshold  $h$ . (c) The upper bound  $\bar{\alpha}_1$  and the probability  $\bar{\mathbb{P}}_{md}(T_{FMA})$  obtained by Monte Carlo simulation as functions of  $\bar{\alpha}_0$  and  $\bar{\mathbb{P}}_{fa}(T_{FMA})$ .

Let us consider the following example of the TCD problem (6.1):  $L=6$ ,  $\lambda_0 = 1$ ,  $(\lambda_1, \dots, \lambda_L) = (5, 2, 5, 6, 0.1, 0.1)$ ,  $m_x = 20$ . As in the case of Gaussian variance, the probabilities  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$  and  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$  are estimated by using a  $10^6$ -repetition Monte Carlo simulation. To illustrate the sharpness of the upper bounds  $\bar{\alpha}_0^{\text{w.a.}}$  and  $\bar{\alpha}_1$ , their comparison with the probabilities  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$  and  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$  estimated by simulation is shown in Figure 4. It follows that the bounds  $\bar{\alpha}_0^{\text{w.a.}}$  and  $\bar{\alpha}_1$  converge rather quickly to the probabilities of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$  and missed detection  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$ .

## 7. DISCUSSION

Let us give the following interpretations for the key findings of the article.

- If the LLRs are associated r.v.s in the prechange mode, both upper bounds for the FMA operating characteristics (3.7) and (3.10) are valid and, moreover, they are asymptotically equivalent (see equation (3.11)), but for nonasymptotic values of  $\bar{\alpha}_0$  (close to one), the difference between two bounds can be significant and it is preferable to use the previously developed upper bound (3.10).
- As it follows from Theorem 3.1, the inequality  $1 \leq m_x \leq L$  results in a more conservative upper bound for the worst-case probability of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_x)$ . However, this more conservative upper bound is obviously valid for any relation between  $m_x$  and  $L$ ; for example, when  $m_x > L \geq 1$  but the inequality  $\tilde{p} < \frac{m_x}{2(m_x - L)}$  is not satisfied.
- The numerical analysis of three particular sequential TCD problems, presented in Sections 4, 5, and 6, shows that the second term  $(m_x - L)^+ p^2$  in (3.3) improves the sharpness of the upper bound for relatively small values of threshold  $h$ .
- Sometimes it is necessary to consider a more sophisticated generative model (1.1) that includes a known profile  $\theta_{0,n}$  in the prechange mode (instead of the constant parameter  $\theta_0$ ). For the three particular cases of transient change detection considered in the article—that is, in the case of the Gaussian mean, variance, and the parameter of exponential distribution—the known profile of  $\theta_{0,n}$  in the prechange mode does not change the problem. This new generative model can be easily reduced to (1.1) by the transformation of observations  $\tilde{y}_n = y_n - \theta_{0,n}$ ,  $\tilde{y}_n = \frac{y_n}{\sigma_{0,n}}$  and  $\tilde{y}_n = \lambda_{0,n} y_n$ . In the case of arbitrary pre- and postchange distributions with a known prechange profile  $\theta_{0,n}$ , the FMA test can be defined but the calculation of the probabilities  $\bar{\mathbb{P}}_{\text{md}}(T)$  and  $\bar{\mathbb{P}}_{\text{fa}}(T; m_x)$  is still open. Finally, the case of a slowly varying prechange profile  $\theta_{0,n}$  with an adaptive FMA test was studied in the framework of monitoring a nuclear reactor in Nikiforov et al. (2020).
- Some practical examples of the transient change detection (with dynamic profiles) can be found in the literature (Guépié, Fillatre, and Nikiforov 2012b, 2012c; Noumir et al. 2012; Do, Fillatre, and Nikiforov 2015; Do et al. 2017; Nikiforov et al. 2020; Tartakovsky et al. 2021; Mana, Guépié, Deprost et al. 2022; Mana, Guépié, and Nikiforov 2022). Drinking water quality monitoring was considered in Guépié, Fillatre, and Nikiforov (2012a, 2012b, 2012c) and Noumir et al. (2012) and air pollution monitoring was considered in Mana, Guépié, Deprost et al. (2022) and Mana, Guépié, and Nikiforov (2022). The papers by Do, Fillatre, and Nikiforov



(2015) and Do et al. (2017) are devoted to the detection of cyber/physical attacks on SCADA systems, Guépié et al. (2020) and Nikiforov et al. (2020) are devoted to monitoring of a sodium-cooled fast reactor and its heat exchanger, and Tartakovsky et al. (2021) is devoted to extracting streaks of faint space objects with unknown orbits from digital frames, captured with ground-based telescopes.

Finally, the following three themes can be considered for further research. First, as mentioned in Sections 1 and 2, the current article considers the optimization of the truncated SPRT tests (reduced to the FMA test) without limiting requirement that the LLRs are associated random variables. The (asymptotic) optimality in the general class of sequential tests is still open. The second theme is the case of an unknown duration of transient changes, which is upper bounded by the required time to alert. The third theme is the case of an unknown profile shape with some known (or at least bounded) average statistical characteristics of the postchange distribution(s) (like the energy of the postchange signal).

## 8. CONCLUSION

The problem of the sequential TCD of an arbitrary transient change profile was considered in this article. In contrast to the previous study, we relax the assumption that the LLRs of the observations are associated r.v.s in the prechange mode. Hence, the verification that the LLRs are associated r.v.s is skipped and the transient change profile can be arbitrary. By using a new upper bound for the worst-case probability of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_\alpha)$ , it was established that the optimization of the WL CUSUM test with a variable threshold leads to the FMA test. Three particular sequential TCD problems—the Gaussian mean case, the Gaussian variance case, and the exponential distribution case—were examined to verify the sharpness of the upper bounds for the FMA test operating characteristics. It follows that the obtained upper bounds converge rather quickly to the exact worst-case probabilities of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}}; m_\alpha)$  and missed detection  $\bar{\mathbb{P}}_{\text{md}}(T_{\text{FMA}})$ , computed by numerical integration or by using the Monte Carlo simulation, when the FMA test threshold  $h$  increases.

## APPENDIX A. PROOF OF THEOREM 3.1

To prove Theorem 3.1, we begin with the first assertion of theorem 2 of Guépié, Fillatre, and Nikiforov (2017); see (3.2). Let us rewrite (3.2) as follows:

$$\bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_\alpha) = \mathbb{P}_0(L \leq T_{\text{WL}} < L + m_\alpha) = \mathbb{P}_0\left(\bigcup_{n=L}^{L+m_\alpha-1} \left\{ \max_{1 \leq k \leq L} [Z_{n-k+1}^n - h_k] \geq 0 \right\}\right). \quad (\text{A.1})$$

The worst-case probability of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_\alpha)$  for a given prechange period  $m_\alpha$  is upper bounded by using the improved Boole-Bonferroni inequality for events  $A_1, \dots, A_n$  (Worsley 1982; Lin and Bai 2010):

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i=2}^n \mathbb{P}(A_1 \cap A_i). \quad (\text{A.2})$$

From (A.1) and (A.2), we get

$$\begin{aligned} \bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_x) &\leq \sum_{n=L}^{L+m_x-1} \mathbb{P}_0 \left( \max_{1 \leq k \leq L} [Z_{n-k+1}^n - h_k] \geq 0 \right) \\ &\quad - \sum_{n=L+1}^{L+m_x-1} \mathbb{P}_0 \left( \left\{ \max_{1 \leq k \leq L} [Z_{L-k+1}^L - h_k] \geq 0 \right\} \cap \left\{ \max_{1 \leq k \leq L} [Z_{n-k+1}^n - h_k] \geq 0 \right\} \right). \end{aligned} \quad (\text{A.3})$$

Let us consider two cases.

The first case is  $1 \leq m_x \leq L$ . Here, all events in the second term  $\mathbb{P}(A_i \cap A_j)$  on the right-hand side improving the Boole-Bonferroni inequality (A.2) are dependent. Therefore, in this case, only the first term in inequality (A.3) is used (pure Boole's inequality). Because the r.v.s  $Z_{n-k+1}^n$ ,  $n = L, \dots, L + m_x - 1$ , are equidistributed under the prechange measure for the same given  $k = 1, \dots, L$ , we get

$$\bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_x) \leq m_x p, \quad p = \mathbb{P}_0 \left( \max_{1 \leq k \leq L} [Z_{L-k+1}^L - h_k] \geq 0 \right). \quad (\text{A.4})$$

The second case is  $m_x > L \geq 1$ . Due to the independence of the r.v.s  $Z_1^L, \dots, Z_L^L$  of the r.v.s  $Z_{L+1}^{2L}, \dots, Z_{2L}^{2L}, \dots$ , it follows from (A.3) that

$$\begin{aligned} \bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_x) &\leq \sum_{n=L}^{L+m_x-1} \mathbb{P}_0 \left( \max_{1 \leq k \leq L} [Z_{n-k+1}^n - h_k] \geq 0 \right) \\ &\quad - \sum_{n=2L}^{L+m_x-1} \mathbb{P}_0 \left( \max_{1 \leq k \leq L} [Z_{L-k+1}^L - h_k] \geq 0 \right) \cdot \mathbb{P}_0 \left( \max_{1 \leq k \leq L} [Z_{n-k+1}^n - h_k] \geq 0 \right). \end{aligned} \quad (\text{A.5})$$

Because the r.v.s  $Z_{n-k+1}^n$  and  $Z_{L-k+1}^L$  are equidistributed under the prechange measure for the same given  $k = 1, \dots, L$ , we get from (A.5) by using Boole's inequality

$$\bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_x) \leq m_x p - (m_x - L)p^2, \quad p = \mathbb{P}_0 \left( \max_{1 \leq k \leq L} [Z_{L-k+1}^L - h_k] \geq 0 \right). \quad (\text{A.6})$$

Finally, putting together (A.6) and (A.4), we get

$$\bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_x) \leq \min\{1, m_x p - (m_x - L)^+ p^2\}. \quad (\text{A.7})$$

## APPENDIX B. PROOF OF THEOREM 3.2

Let us establish the upper bound for the probability of false alarm  $\bar{\mathbb{P}}_{\text{fa}}(T_{\text{FMA}})$  given in (3.5). As before, two cases should be considered.

The first case is  $1 \leq m_x \leq L$ . It follows from Theorem 3.1 (by using Boole's inequality) that

$$\begin{aligned} \bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_x) &\leq m_x \mathbb{P}_0 \left( \max_{1 \leq k \leq L} [Z_{L-k+1}^L - h_k] \geq 0 \right) \\ &\leq H(h_1, \dots, h_L) = m_x \left[ \sum_{k=1}^{L-1} \mathbb{P}_0 \left( Z_{L-k+1}^L \geq h_k \right) + 1 - F_{Z,L}(h_L) \right], \end{aligned} \quad (\text{B.1})$$

which establishes the first part of (3.5). Let us consider the following equation  $\bar{\alpha}_0 = H(h_1, \dots, h_L)$ . It is easy to see that

$$F_{Z,L}(h_L) = 1 - \frac{\bar{\alpha}_0}{m_x} + \sum_{k=1}^{L-1} \mathbb{P}_0 \left( Z_{L-k+1}^L \geq h_k \right) \quad (\text{B.2})$$

provided that the thresholds  $h_1, \dots, h_{L-1}$  are so chosen that the right-hand side of the equation belongs to the open interval  $]0, 1[$ . Because the function  $x \mapsto F_{Z,L}(x)$  is continuous nondecreasing,

the equation  $F_{Z,L}(x) = \beta$  has a solution for any prescribed constant  $\beta \in ]0, 1[$ . Hence, the threshold  $h_L$  is a solution of the following equation:

$$h_L = F_{Z,L}^{-1} \left( 1 - \frac{\bar{\alpha}_0}{m_x} - \sum_{k=1}^{L-1} \mathbb{P}_0 \left( Z_{L-k+1}^L \geq h_k \right) \right) \quad (\text{B.3})$$

and

$$\bar{\alpha}_1(h_1, \dots, h_{L-1}) = G \left[ F_{Z,L}^{-1} \left( 1 - \frac{\bar{\alpha}_0}{m_x} - \sum_{k=1}^{L-1} \mathbb{P}_0 \left( Z_{L-k+1}^L \geq h_k \right) \right) \right]. \quad (\text{B.4})$$

The goal of the minimization problem (3.4) is to minimize the value of  $\bar{\alpha}_1$  for any given  $\bar{\alpha}_0 \in ]0, 1[$ . The function  $x \mapsto G(x)$  is nondecreasing and the function  $x \mapsto F_{Z,L}^{-1}(x)$  is increasing. Therefore, the optimal solution is reached when  $h_1 \rightarrow \infty, \dots, h_{L-1} \rightarrow \infty$  and

$$h_L^{\text{w.a.}} = h_L^* = \lim_{h_1 \rightarrow \infty, \dots, h_{L-1} \rightarrow \infty} h_L(h_1, \dots, h_{L-1}) = F_{Z,L}^{-1} \left( 1 - \frac{\bar{\alpha}_0}{m_x} \right) \quad (\text{B.5})$$

and it is equal to

$$\bar{\alpha}_1 = \lim_{h_1 \rightarrow \infty, \dots, h_{L-1} \rightarrow \infty} \bar{\alpha}_1(h_1, \dots, h_{L-1}) = G \left[ F_{Z,L}^{-1} \left( 1 - \frac{\bar{\alpha}_0}{m_x} \right) \right] \quad (\text{B.6})$$

for any given  $\bar{\alpha}_0 \in ]0, 1[$ . Hence, the first part of (3.6) and (3.7) is established.

The second case is  $m_x > L \geq 1$ . Let us first consider the following quadratic function  $f(x) = m_x x - (m_x - L)x^2$ . The maximum of this function is reached at  $x_0 = \frac{m_x}{2(m_x - L)}$ . It is easy to see that  $x_0 = \frac{m_x}{2(m_x - L)} \geq \frac{1}{2}$  for any  $m_x > L \geq 1$ . Taking into account that the function  $x \mapsto f(x)$  is increasing for  $x \leq x_0$  and putting together (A.6) and Boole's inequality, we get the following majoration:

$$\bar{\mathbb{P}}_{\text{fa}}(T_{\text{WL}}; m_x) \leq m_x p - (m_x - L)p^2 \leq H(h_1, \dots, h_L) = m_x \tilde{p} - (m_x - L)^+ \tilde{p}^2, \quad (\text{B.7})$$

where  $p = \mathbb{P}_0(\max_{1 \leq k \leq L} [Z_{L-k+1}^L - h_k] \geq 0)$ ,  $\tilde{p} = \sum_{k=1}^{L-1} \mathbb{P}_0(Z_{L-k+1}^L \geq h_k) + 1 - F_{Z,L}(h_L)$  and assuming that  $\tilde{p} < \frac{m_x}{2(m_x - L)}$ . The second part of (3.5) is established. Putting together (B.1) and (B.7), we obtain (3.9).

Let us consider the equation  $\bar{\alpha}_0 = H(h_1, \dots, h_L)$ . A suitable solution of this equation

$$\tilde{p}_1 = \frac{m_x - \sqrt{m_x^2 - 4(m_x - L)\bar{\alpha}_0}}{2(m_x - L)} < \frac{m_x}{2(m_x - L)} \quad (\text{B.8})$$

always exists because  $m_x^2 - 4(m_x - L)\bar{\alpha}_0 \geq 0$  for any  $m_x > L \geq 1$ . By the same reason as in the first part of this proof, we get

$$F_{Z,L}(h_L) = 1 - \frac{m_x - \sqrt{m_x^2 - 4(m_x - L)\bar{\alpha}_0}}{2(m_x - L)} + \sum_{k=1}^{L-1} \mathbb{P}_0 \left( Z_{L-k+1}^L \geq h_k \right). \quad (\text{B.9})$$

Hence, the threshold  $h_L$  is a solution of the following equation:

$$h_L = F_{Z,L}^{-1} \left( 1 - \frac{m_x - \sqrt{m_x^2 - 4(m_x - L)\bar{\alpha}_0}}{2(m_x - L)} + \sum_{k=1}^{L-1} \mathbb{P}_0 \left( Z_{L-k+1}^L \geq h_k \right) \right) \quad (\text{B.10})$$

and

$$\bar{\alpha}_1(h_1, \dots, h_{L-1}) = G \left[ F_{Z,L}^{-1} \left( 1 - \frac{m_x - \sqrt{m_x^2 - 4(m_x - L)\bar{\alpha}_0}}{2(m_x - L)} + \sum_{k=1}^{L-1} \mathbb{P}_0 \left( Z_{L-k+1}^L \geq h_k \right) \right) \right]. \quad (\text{B.11})$$

By the same reason as in the first part of this proof, the optimal solution of the minimization problem (3.4) is reached when  $h_1 \rightarrow \infty, \dots, h_{L-1} \rightarrow \infty$

$$\begin{aligned} h_L^{\text{w.a.}} = h_L^* &= \lim_{h_1 \rightarrow \infty, \dots, h_{L-1} \rightarrow \infty} h_L(h_1, \dots, h_{L-1}) \\ &= F_{Z,L}^{-1} \left( 1 - \frac{m_x - \sqrt{m_x^2 - 4(m_x - L)\bar{\alpha}_0}}{2(m_x - L)} \right) \end{aligned} \quad (\text{B.12})$$

and it is equal to

$$\begin{aligned} \bar{\alpha}_1 &= \lim_{h_1 \rightarrow \infty, \dots, h_{L-1} \rightarrow \infty} \bar{\alpha}_1(h_1, \dots, h_{L-1}) \\ &= G \left[ F_{Z,L}^{-1} \left( 1 - \frac{m_x - \sqrt{m_x^2 - 4(m_x - L)\bar{\alpha}_0}}{2(m_x - L)} \right) \right] \end{aligned} \quad (\text{B.13})$$

for any given  $\bar{\alpha}_0 \in ]0, 1[$ . Hence, the second part of (3.6) and also (3.7) and (3.8) is established.

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## ORCID

Fatima Ezzahra Mana  <http://orcid.org/0000-0002-9790-3964>

Blaise Kévin Guépié  <http://orcid.org/0000-0002-0241-7315>

Igor Nikiforov  <http://orcid.org/0000-0001-7051-9308>

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