A decomposition and coordination method for optimising (Q , S) policies in a two-echelon distribution system with joint replenishment

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A continuous-review, two-echelon distribution system with one central distribution centre (CDC) and multiple regional distribution centres (RDCs) is studied. The CDC jointly replenishes its inventories of multiple items from an external supplier, while each RDC replenishes its inventories of the items from the CDC. Each RDC faces a Poisson demand for each item, and the inventories of each stock in the system are controlled by a (Q, $\boldsymbol{S}$) policy. Under this policy, an order is placed by a stock whenever its aggregate demand since the last order reaches a given quantity, and the inventory position of each item is raised up to its order-up-to level after the order placement. The objective is to optimise these (Q, $\boldsymbol{S}$) policies so that the expected total cost of this system is minimised. We propose a decom-position and coordination method for this optimisation after deriving analytically the cost function of the system. Our extensive numerical experiments demonstrate the effectiveness of the proposed method. Furthermore, a parameter sensitivity analysis is conducted to analyse the impacts of some key system parameters on the performance of the method, and managerial insights are provided for optimising distribution systems with joint replenishment and real applications of the method.

1. Introduction

Inventory management is crucial in supply chains, especially in distribution systems. This paper focuses on optimising inventory policies for a two-echelon distribution system with joint replenishment, inspired by a real inventory problem in Alibaba's supply chain. This system comprises a central distribution centre (CDC) and multiple regional distribution centres (RDCs). Inventories are continuously reviewed at the CDC and each RDC. The CDC replenishes its inventories of multiple items from an outside supplier and then supplies the RDCs. Each RDC faces a Poisson demand for each item. The inventories of multiple items are jointly replenished using a continuous-review (Q, $\boldsymbol{S}$) policy at each stock. Under this policy, an order is placed whenever the aggregate demand of a stock since the last order reaches a given quantity Q, and the inventory position of each item is raised to its order-up-to level specified by $\boldsymbol{S}$ after the order placement. The lead time for inventory replenishment from the outside supplier to the CDC is assumed constant, whereas the lead time from the CDC to each RDC is stochastic because of possible stockout at the CDC. Each joint replenishment incurs a major ordering cost and minor ordering costs. The major ordering cost corresponds to the major expenses of placing a bulk order of multiple items. It may include setup costs, transportation costs, and administrative costs. The minor ordering cost of each item represents the minor expenses incurred when ordering this item in a bulk order. The minor ordering cost is item-specific Holding costs are charged at the CDC and RDCs, while excess demand at the RDCs results in backlogged orders and associated shortage costs. The problem is to optimise the (Q, $\boldsymbol{S}$) policies to minimise the long-run average cost per time unit of the system.

Joint inventory replenishment can realise substantial cost savings by coordinating and combining replenishments of multiple items (Wang and Chen 2022). However, existing studies either focus on multi-echelon distribution systems with a single item or single stocks with multiple items. To the best of our knowledge, the problem we consider was never studied in the literature.

In practice, the inventory policy for each stock in a distribution system is often determined independently, without considering other stocks. However, this may result in suboptimal system-wide inventory policies. Multi-echelon inventory optimisation aims to find coordinated inventory policies that globally optimise the system. Incorporating joint replenishments into this opti-

misation can enhance performance and reduce costs for the system. This is also true for a multi-item lot-sizing model with individual inventory costs and joint production costs (Leopoulos and Proth 1985). However, optimising multi-echelon inventory systems with joint replenishments is very challenging. Even for a multi-echelon inventory system with a single item, the structure of its optimal echelon stock policy is unknown (Dolgui and Proth 2010). To the best of our knowledge, neither exact nor heuristic algorithm for solving this problem was reported in the literature.

With respect to the literature, our contribution to the study of stochastic inventory systems with joint replenishments is fourfold:

1. We study for the first time the optimisation of (Q, $\boldsymbol{S}$) policies for a two-echelon joint replenishment distribution system and propose an effici t and effective decomposition and coordination approach.
2. We derive analytically the cost functions of all stocks in the distribution system.
3. We conduct extensive numerical experiments to evaluate the performance of the proposed approach.
4. We conduct a sensitivity analysis to evaluate the impacts of some key parameters on the system's performance and provide managerial insights for optimising distribution systems with joint replenishments and real applications of the approach.

The rest of this paper is organised as follows. Section 2 provides a review of the literature related to our work. The two-echelon distribution system studied is described in Section 3 with analytical formulation of its cost functions. Section 4 presents a decomposition and coordination algorithm for optimising the (Q, $\boldsymbol{S}$) policies of this system. The quality of the (Q, $\boldsymbol{S}$) policies found by the algorithm is evaluated by an error bound in Section 5. Section 6 presents numerical experiments of the algorithm and a sensitivity analysis with managerial insights provided. Section 7 concludes this paper with perspectives for future research.

2. Literature review

Our work contributes to the literature on stochastic multi-echelon distribution and joint replenishment inventory systems. Andersson, Axsäter, and Marklund (1998) and Andersson and Marklund (2000) studied a single-item continuous-review two-echelon distribution system controlled by (R, Q) policies, and proposed a decomposition and coordination approach for their optimisation. Berling and Marklund (2014) extended their previous work by addressing lead-time variability. However, these models overlook the joint replenishment of multiple items, a common practice in real distribution systems. Our study extends Andersson and Marklund's approach to consider joint replenishments controlled by (Q, S) policies. For a comprehensive review of the literature on stochastic multi-echelon inventory systems, please refer to De Kok et al. (2018).

Joint Replenishment Problems (JRPs) have been an active research topic since multiple products are usually replenished in real warehouses and retail stores. The joint replenishment of multiple items can reduce total logistics costs by sharing ordering and transportation costs. Existing literature on JRPs can be classifie into deterministic JRPs and stochastic JRPs (SJRPs). Previous studies on JRPs were reviewed by Khouja and Goyal (2008) and Peng, Wang, and Wang (2022).

Various policies have been proposed for joint replenishments. ($\boldsymbol{s}$, $\boldsymbol{c}$, $\boldsymbol{S}$) policy and (Q, $\boldsymbol{S}$) policy are two typical continuous review joint replenishment policies. Under ($\boldsymbol{s}$, $\boldsymbol{c}$, $\boldsymbol{S}$) policy, whenever any item's inventory position is lower than its must-order level s, a joint replenishment order will be placed to raise its inventory position to its order-up-to level S, and other items if their inventory position is below its can-order level c will also be included in this order. In our study, we adopt (Q, $\boldsymbol{S}$) policy, also known as QS policy introduced by Renberg and Planche (1967) and studied in Cachon (2001). Under (Q, $\boldsymbol{S}$) policy, a joint replenishment order is placed to bring each item's inventory position to its order-up-to level S when the total demand of all items since the last order reaches a specified quantity Q. Golany and Lev-Er (1992) conducted extensive simulations comparing (s, c, S) policy and (Q, $\boldsymbol{S}$) policy and identifie limitations of (s, c, S) policy. Li and Schmidt (2020) also compared the two policies numerically and showed that no policy is better than the other in all instances.

Most papers on multi-echelon joint replenishment inventory systems assume deterministic demands because of much higher complexity of their stochastic counterpart. Even for a deterministic multi-echelon joint replenishment system, findi g its optimal inventory policy is quite difficult, so metaheuristics were widely applied in its optimisation, such as genetic algorithm (Cha, Moon, and Park 2008), Tabu search (Liu et al. 2018), and differential evolution (Wang et al. 2020). Padiyar et al. (2022) studied a multi-echelon supply chain with deteriorating items, imperfect production, and uncertain inflation. Das, Baran Hui, and Jain (2019) presented a probabilistic model for optimising inventory policies in a multi-echelon distribution system with multiple products. Although the two papers also deal with multi-echelon inventory systems with joint replenishment, Padiyar et al. studied a distribution system with

Table 1. Comparison with related literature.

Authors	Review type	Demand type	Inventory Policy	Method
Cha, Moon, and Park (2008)	*P*	Deterministic	Periodic JR	Genetic algorithms
Liu et al. (2018)	*P*	Deterministic	Periodic JR	Tabu search algorithm
Wang et al. (2020)	*P*	Deterministic	Periodic JR	Differential evolution algorithms
Zhou, Chen, and Ge (2013)	*P*	Poisson	(*T*, ***S***)	Genetic algorithms
This paper	C	Poisson	(*Q*, ***S***)	A decomposition and coordination method

C: continuous review; *P*: periodic review. Periodic JR: periodic joint replenishment.

deterministic demands and Das et al. considered an integer policy of ordering, whereas we consider a distribution system with stochastic demands and (*Q*, ***S***) policy for joint replenishment.

Our paper considers a stochastic two-echelon distribution system with Poisson demands controlled by continuous review (*Q*, *S*) policies. To the best of our knowledge, the only work considering a similar system is that of Zhou, Chen, and Ge (2013), who considered a multi-item, multi-echelon distribution system with Poisson demands and controlled by (*T*, ***S***) policies. They proposed a genetic algorithm for optimising the (*T*, ***S***) policies. Compared with time-based (*T*, ***S***) policies, quantity-based (*Q*, ***S***) policies are more difficult to optimise in a multi-echelon inventory system with joint replenishments. The challenges we have to tackle include analytical evaluation of expected total cost of the system due to the coupling among multiple items and the difficulty of coordinating replenishments across the items. Instead of proposing a metaheuristic algorithm, we develop an iterative decomposition and coordination approach for optimising the (*Q*, ***S***) policies in a two-echelon distribution system based on approximate formulation of its cost functions. This approachcan evaluate the quality of the inventory policies obtained. Table 1 summarises the related works on multi-echelon joint replenishment inventory systems.

3. Problem description and preliminaries

The two-echelon distribution system and its inventory policy optimisation problem we study is described in this section, with the assumptions made and the notations used in this paper. In addition, the determination of order quantities for each stock is discussed, followed by the presentation of cost functions used in both an exact and approximate system model.

3.1. The two-echelon distribution system

We consider a two-echelon distribution system consisting of a central distribution centre (CDC) and multiple regional distribution centres (RDCs) as shown in Figure 1. The CDC sources various items from external suppliers, and each RDC replenishes these items from the

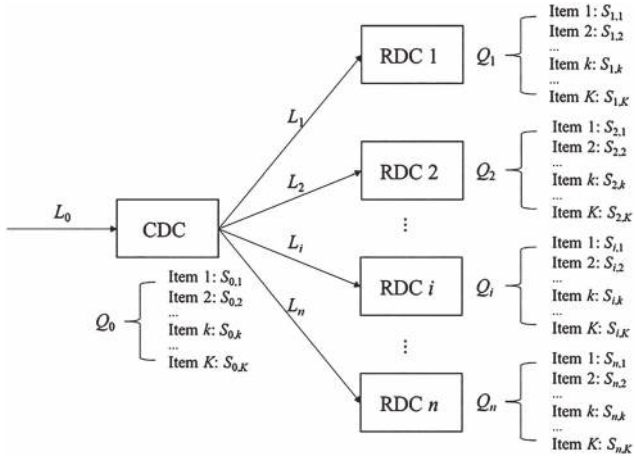

Figure 1. A two-echelon distribution system with one CDC and multiple RDCs.

CDC. The system adopts a continuous-review (*Q*, ***S***) policy for each stock, where an order is placed by each stock to raise its inventory position to its order-up-to level *S* when its aggregate demand since the last order reaches quantity *Q*. The replenishment lead time of the CDC is assumed constant, but the replenishment lead time of each RDC is stochastic due to potential stockouts at the CDC.

3.2. Assumptions and notations

The assumptions made and the notations used in this paper are listed in the following.

Assumptions

(1) Multiple items are involved at the CDC and each RDC.
(2) The inventory status of each item at the CDC and each RDC is continuously reviewed.
(3) The demand of each item observed by each RDC is subject to a Poisson distribution. This demand is independent of any other item observed by the same RDC and the demand of any item observed by any other RDC.
(4) All items are jointly replenished at the CDC and each RDC.
(5) There is a common transportation (delivery) time for all items in each joint replenishment at each RDC

and a constant replenishment lead time for all items in each joint replenishment at the CDC.

(6) Any joint replenishment order is never partially fulfill d. If the CDC does not have enough on-hand inventory for some items to fully fulfil a joint replenishment order of an RDC, it will wait until it has suffici t inventory to fully fulfi this order.

(7) If multiple joint replenishment orders from the RDCs are waiting for fulfilment at the CDC, these orders are fulfil ed by applying the **FCFS** rule.

(8) The inventory of each item at the external supplier is ample at any time.

Notations

Sets and indices

N: set of all RDCs in the distribution system, $N = 1, 2, \ldots, N$, where N is the number of RDCs.

$N \cup 0$: set of all stocks in the distribution system, where 0 represents the CDC.

K: set of all items involved in the distribution system, $K = 1, 2, \ldots, K$, where K is the number of items.

i: RDC index, $i \in N$.

k: item index, $k \in K$.

t: time index or length of time.

Parameters

L_0: constant replenishment lead-time of the CDC.

l_i: constant transportation time from the CDC to RDC i.

L_i: lead-time for an order placed by RDC i to be shipped by the CDC, a stochastic variable.

$\bar{L}_i$: expected lead time for each order placed by RDC i to be shipped by the CDC, $\bar{L}_i = E[L_i]$.

A_0: major ordering cost for each joint replenishment of the CDC.

$a_{0,k}$: minor ordering cost of item k for each joint replenishment of the CDC.

A_i: major ordering cost for each joint replenishment of RDC i.

$a_{i,k}$: minor ordering cost of item k for each joint replenishment of RDC i.

$D_{i,k}^t$: Poisson demand of item k observed by RDC i during a time interval of length t.

D_i^t: total demand of all items observed by RDC i during a time interval of length t.

$\lambda_{i,k}$: mean demand rate of item k at RDC i, $\lambda_{i,k}t = E[D_{i,k}^t]$.

Λ_i: total demand rate of all items at RDC i, $\Lambda_i = \sum_{k=1}^{K} \lambda_{i,k}$.

$f_{i,k}(x,t)$: probability density function (p.d.f) of Poisson random variable $D_{i,k}^t$.

$F_{i,k}(x,t)$: cumulative distribution function (c.d.f) of $D_{i,k}^t$.

$h_{0,k}$: holding cost per unit of item k per time unit at the CDC.

$h_{i,k}$: holding cost per unit of item k per time unit at RDC i. It is assumed that $h_{i,k} > h_{0,k}$ for any k. That is, $h'_{i,k} = h_{i,k} - h_{0,k} > 0$ for any k, $h'_{i,k}$ is an increment of unit holding cost.

$p_{i,k}$: shortage cost per unit of item k per time unit at RDC i.

Decision variables

Q_i: joint replenishment order quantity of RDC i.

Q_0: joint replenishment order quantity of the CDC.

$S_{i,k}$: order-up-to level of item k at RDC i.

S_i: $S_i = (S_{i,k}, k \in K)$, order-up-to level vector of RDC i.

$S_{0,k}$: order-up-to level of item k at the CDC.

S_0: $S_0 = (S_{0,k}, k \in K)$, order-up-to level vector of the CDC.

Other variables

s_i: aggregate reorder point of RDC i, $s_i = \sum_{k=1}^{K} S_{i,k} - Q_i$.

s_0: aggregate reorder point of the CDC, $s_0 = \sum_{k=1}^{K} S_{0,k} - Q_0$.

s_0^{E*}: optimal aggregate reorder point of the CDC in an exact model.

s_0^{A*}: optimal aggregate reorder point of the CDC in an approximate model.

C_0: expected cost of the CDC per time unit.

C_i^E: expected cost of RDC i per time unit in an exact model.

TC^{E}: expected total cost of the distribution system per time unit in the exact model.

C_i^A: expected cost of RDC i per time unit in the approximate model.

TC^{A}: expected total cost of the distribution system per time unit in the approximate model.

Inventory status variables

$I_{0,k}(t)$: on-hand inventory of item k at the CDC at time t.

$I_0(t)$: aggregate on-hand inventory of all items at the CDC at time t, $I_0(t) = \sum_{k=1}^{K} I_{0,k}(t)$.

$I_{i,k}(t)$: on-hand inventory of item k at RDC i at time t.

$I_i(t)$: aggregate on-hand inventory of all items at RDC i at time t, $I_i(t) = \sum_{k=1}^{K} I_{i,k}(t)$.

$B_0(t)$: aggregate backorders of all items at the CDC at time t.

$B_i(t)$: aggregate backorders of all items at RDC i at time t.

3.3. Determination of order quantity for each (Q, S) policy in the distribution system

Optimising the reorder point and order quantity for each stock in a distribution system with multiple items is challenging. In the literature, the values of these parameters are typically determined sequentially. The order quantities for all stocks are first determined, followed by optimising the reorder points. Chen and Zheng (1997) indicated that a suitable deterministic model for determining the order quantities in a stochastic inventory system minimises its expected total cost, as long as the reorder points are well optimised. Built on this insight, Andersson and Marklund (2000) optimised the reorder points in a distribution system by predefini g the order quantities. Following this sequential approach, we solve a deterministic model to determine the joint replenishment order quantities for all stocks in the considered distribution system. Subsequently, we optimise the order-up-to levels S for all items at all stocks by employing a method to be introduced in Section 4. The determination of the order quantity Q for each (Q, $\boldsymbol{S}$) policy in the system is given in Appendix A.

3.4. Cost functions in exact and approximate models

We consider two models, an exact model and an approximate model, for evaluating the expected total cost of the distribution system. This cost is the sum of the expected costs of the CDC and the RDCs. In the exact model, the replenishment lead-time of each RDC is a stochastic variable, whereas this stochastic lead time is replaced by its mean value in the approximate model. In the approximate model, the expected cost of the CDC is also exactly calculated, only the expected cost of each RDC is approximately calculated. Since all items are jointly replenished at the CDC and each RDC, as soon as the joint replenishment order quantity Q is determined for each (Q, $\boldsymbol{S}$) policy in the distribution system, its order-up-levels of all items at each stock will not affect major and minor ordering costs. Therefore, we only need to consider holding costs at the CDC and both holding and shortage costs at each RDC. Note that shortage costs are not charged at the CDC as in most models in the literature.

Schematically, the expected total cost of the distribution system in the exact model can be expressed as:

$$TC^E = C_0 + \sum_{i=1}^{N} C_i^E = C_0(S_0) + \sum_{i=1}^{N} C_i^E(S_i|L_i(S_0)) \quad (1)$$

where TC^{E} is the exact expected total system cost per time unit, C_0 is the exact expected holding cost per time unit of the CDC, C_i^E is the exact expected holding and shortage cost per time unit of RDC i, and $L_i(S_0)$ is the stochastic lead time of RDC i, which depends on the order-up-to level of each item k at the CDC, $k \in K$.

On the other hand, the expected total cost of the system in the approximate model can be expressed as:

$$TC^A = C_0 + \sum_{i=1}^{N} C_i^A = C_0(S_0) + \sum_{i=1}^{N} C_i^A(S_i|\bar{L}_i(S_0)) \quad (2)$$

where TC^{A} is the approximate expected total system cost per time unit, and C_i^A is the approximate expected holding and shortage cost per time unit of RDC i, $\bar{L}_i(S_0)$ is the mean value of $L_i(S_0)$.

An important characteristic of the system is the stochastic Poisson demand faced by each RDC for each item and the independence between any two Poisson demands in this system. The two properties allow us to analytically calculate all costs in equations (1) and (2). See Appendix B for the derivation of the cost functions.

4. Decomposition and coordination

Due to the coupling among items, the coordination of inventory replenishments across K items (K > 1) in a distribution system is more difficult than solving its K single-item inventory replenishment subproblems. In addition, because of the stochastic nature of the lead time $L_i(S_0)$, finding an exact method to calculate the exact expected total cost of the distribution system given by (1) is very difficult. Because of the two reasons, we propose an iterative decomposition and coordination approach to minimise the approximate expected total cost TC^A given by (2). This is motivated by a procedure proposed by Andersson and Marklund (2000) for optimising inventory policies of a distribution system with a single product. We extend their procedure to a distribution system with joint replenishment. In

each iteration of our approach, the approximate model with cost function (2), a complicated stochastic multi-echelon joint replenishment problem, is decomposed into multiple single-item single-stock inventory subproblems. The solutions of these subproblems are coordinated through iteratively adjusting a set of induced shortage costs per unit of item per unit of time introduced for backorders at the CDC, so that they gradually approach an optimal or a near-optimal solution of the model.

4.1. Decomposition of the approximate model

Although shortage costs are only charged at the RDCs, the inventory decisions (the order-up-to levels) of the CDC have great impacts on the shortage costs and holding costs of each RDC, because the upstream CDC and the downstream RDCs are coupled through stochastic lead times $L_i(S_0)$, $i \in N$ between them. To decouple the CDC's inventory decisions from those of the RDCs, we will introduce artificial shortage costs for backorders at the CDC as mentioned above. On the other hand, for the approximate model (2), as soon as the mean lead times $\bar{L}_i(S_0)$, $i \in N$ are known, the inventory decisions of the RDCs can be decoupled from that of the CDC. That is, we can defin the RDC decision problems, one for each RDC, as follows:

$$\min_{S_i} C_i^A(S_i|\bar{L}_i), \qquad i \in N \tag{3}$$

Given the value of $\bar{L}_i$, we can analytically formulate the cost function $C_i^A(S_i|\bar{L}_i)$ and demonstrate that it is item-separable, i.e. $C_i^A(S_i|\bar{L}_i) = \sum_{k=1}^{K} C_{i,k}^A(S_{i,k}|\bar{L}_i)$. See Appendix B for this formulation. After solving the decision problem of each RDC i, we can obtain its optimal solution denoted by $S_i^* = (S_{i,k}^*, k \in K)$, where $S_{i,k}^*$ is a function of $\bar{L}_i$ and $\bar{L}_i$ is in turn a function of $S_0 = (S_{0,k}, k \in K)$. For simplicity, we use $\bar{L}_i$ instead of $\bar{L}_i(S_0)$ in the following formulations in case of no confusion.

To analytically formulate the CDC's expected holding cost $C_0(S_0)$, we need the notion of 'reserved units' introduced in Lee and Moinzadeh (1987) and Andersson and Marklund (2000). Because we assume the CDC adopts the full order fulfilme t policy for RDC orders, the CDC may have reserved units for some items, where a unit available at the CDC is called a reserved unit if it is allocated to an order placed by an RDC but not yet shipped. Let $B_{0,k}^p$ denote the backorder level of item k at the CDC obtained in case of allowing partial fulfil ent of RDC orders, $B_{0,k}^r$ and $B_{0,k}$ denote respectively the number of reserved units and the backorder level of item k at the CDC in case of only shipping complete RDC orders, we have $B_{0,k} = B_{0,k}^p + B_{0,k}^r$. By applying Little's Law, we have:

$$E[B_{0,k}] = \sum_{i=1}^{N} \lambda_{i,k}(\bar{L}_i - l_i) \ , \quad \forall k \in K \tag{4}$$

or equivalently

$$E[B_{0,k}^r] = \sum_{i=1}^{N} \lambda_{i,k}(\bar{L}_i - l_i) - E[B_{0,k}^p] \ , \quad \forall k \in K \tag{5}$$

where $\bar{L}_i - l_i$ is the mean waiting time of each order placed by RDC i at the CDC.

The analytical formulation of $C_0(S_0)$ requires to calculate $E[B_{0,k}^r]$, which can be derived from $E[B_{0,k}^p]$ and $\bar{L}_i$ according to (5). We can also demonstrate that $C_0(S_0)$ is item-separable, i.e. $C_0(S_0) = \sum_{k=1}^{K} C_{0,k}(S_{0,k})$. See Appendix B for this formulation.

Next, we discuss how to calculate $\bar{L}_i$. Let $B_{0,k}^i$ denote the number of reserved units allocated to orders of RDC i plus backorders at the CDC placed by RDC i for item k, we have $B_{0,k} = \sum_{i=1}^{N} B_{0,k}^i$. According to Little's Law, we can obtain:

$$E[B_{0,k}^i] = \lambda_{i,k}(\bar{L}_i - l_i) \tag{6}$$

Denote $E[B_{0,k}^i]$ by $\bar{B}_{0,k}^i$. From Eq. (6) we can obtain:

$$\bar{L}_i = l_i + \frac{\bar{B}_{0,k}^i}{\lambda_{i,k}} \tag{7}$$

Turning to inventory decisions of the CDC, its order-up-to levels $S_{0,k}$, $k \in K$ will affect the shortage costs of each RDC i through the expected lead-time $\bar{L}_i$ or expected backorder levels $\bar{B}_{0,k}^i$, $k \in K$, and thus affect the total cost of the distribution system. To capture this influence, we introduce shortage costs induced by the change of expected lead-time $\bar{L}_i$ in the CDC's cost function when it makes its inventory decisions. Instead of minimising $C_0(S_0)$, the CDC minimises an augmented cost function $\tilde{C}_0(S_0)$ with additional induced shortage costs (see Eq. (9) in the following).

For this purpose, define:

$C_{i,k}^A(S_{i,k}^*|\bar{L}_i)$: minimum holding and shortage cost of item k at RDC i in the approximate model given $\bar{L}_i$.
$C_{i,k}^A(S_{i,k}^*|\bar{B}_{0,k}^i)$: minimum holding and shortage cost of item k at RDC i in the approximate model given $\bar{B}_{0,k}^i$.
$\lambda_{i,k}$: mean demand rate of item k observed by RDC i.
$\beta_{i,k}$: induced shortage cost per unit of item k per time unit for backorders at the CDC ordered by RDC i. $\beta_{i,k} = dC_{i,k}^A(S_{i,k}{}^*|\bar{B}_{0,k}^i)/d\bar{B}_{0,k}^i$, where $\beta_i = (\beta_{i,k}, k \in K)$.

To facilitate the calculation of the first derivative $\beta_{i,k}$, we replace $C^A_{i,k}(S^*_{i,k}|\bar{L}_i)$ by $f(\bar{L}_i)$, replace $C^A_{i,k}(S^*_{i,k}|\bar{B}^i_{0,k})$ by $g(\bar{B}^i_{0,k})$, and replace $\bar{B}^i_{0,k} = \lambda_{i,k}(\bar{L}_i - l_i)$ by $h(\bar{L}_i)$. Then, we have $f(\bar{L}_i) = g(h(\bar{L}_i))$, and $\beta_{i,k}$ can be rewritten as:

$$\begin{aligned}\beta_{i,k} &= \frac{dC^A_{i,k}(S_{i,k}{}^*|\bar{B}^i_{0,k})}{d\bar{B}^i_{0,k}} = \frac{dg(\bar{B}^i_{0,k})}{d\bar{B}^i_{0,k}} \\ &= \frac{dg(h(\bar{L}_i))}{d\bar{L}_i} \cdot \frac{\partial \bar{L}_i}{\partial \bar{B}^i_{0,k}} = \frac{df(\bar{L}_i)}{d\bar{L}_i} \cdot \frac{\partial \bar{L}_i}{\partial \bar{B}^i_{0,k}} \\ &= \frac{dC^A_{i,k}(S_{i,k}{}^*|\bar{L}_i)}{d\bar{L}_i} \cdot \frac{1}{\lambda_{i,k}} \end{aligned} \tag{8}$$

With this equation, the inventory decision problem of the CDC can be formulated as:

$$\begin{aligned}&\min_{S_0} \tilde{C}_0(S_0) \\ &= C_0(S_0) + \sum_{i=1}^{\mathrm{N}} \sum_{k=1}^{\mathrm{K}} \frac{dC^A_{i,k}(S_{i,k}{}^*|\bar{B}^i_{0,k})}{d\bar{B}^i_{0,k}} \cdot \bar{B}^i_{0,k} \\ &= \sum_{k=1}^{\mathrm{K}} C_{0,k}(S_{0,k}) + \sum_{i=1}^{\mathrm{N}} \sum_{k=1}^{\mathrm{K}} \frac{dC^A_{i,k}(S_{i,k}{}^*|\bar{B}^i_{0,k})}{d\bar{B}^i_{0,k}} \cdot \bar{B}^i_{0,k} \\ &= \sum_{k=1}^{\mathrm{K}} C_{0,k}(S_{0,k}) + \sum_{i=1}^{\mathrm{N}} \sum_{k=1}^{\mathrm{K}} \beta_{i,k} \cdot \bar{B}^i_{0,k} \end{aligned} \tag{9}$$

Since each RDC applies a $(Q, \boldsymbol{S})$ policy for its joint replenishment and the value of Q is given, its joint replenishment order process (in terms of order placing instants and orders) observed by the CDC does not depend on the order-up-to levels of the RDCs, but only depends on the demands observed by the CDC and its order-up-levels. For this reason, $\bar{B}^i_{0,k}$ in Eq. (9) can be obtained by discrete event simulation as soon as the CDC's joint replenishment inventory policy is given.

To calculate the coefficient $\beta_{i,k}$ in the decision problem (9), we need to know $\frac{dC^A_{i,k}(S_{i,k}{}^*|\bar{L}_i)}{d\bar{L}_i}$. According to $C^A_{i,k}(S_{i,k}|\bar{L}_i)$ given by Eq. (A.20) in Appendix B, we can derive an analytical expression for this first derivative as follows:

$$\begin{aligned}&\frac{dC^A_{i,k}(S_{i,k}{}^*|\bar{L}_i)}{d\bar{L}_i} \\ &= \frac{1}{Q_i} \sum_{u_i=0}^{Q_i-1} \sum_{m_{i,k}=0}^{u_i} \left(\frac{u_i!}{m_{i,k}!(u_i - m_{i,k})!} \left(\frac{\lambda_{i,k}}{\Lambda_i}\right)^{m_{i,k}} \right. \\ &\quad \left. \times \left(1 - \frac{\lambda_{i,k}}{\Lambda_i}\right)^{u_i - m_{i,k}} \right) \\ &\quad \times \left[\left((S_{i,k} - m_{i,k})(h_{i,k} + p_{i,k}) \right. \right. \\ &\quad \left. \times \left(- \sum_{j=0}^{S_{i,k}-m_{i,k}-1} \frac{1}{j!} \times \theta \right) \right) \\ &\quad - \lambda_{i,k}(h_{i,k} + p_{i,k}) \\ &\quad \times \left(1 - \sum_{j=0}^{S_{i,k}-m_{i,k}-2} \frac{1}{j!} e^{-\lambda_{i,k}\bar{L}_i} (\lambda_{i,k}\bar{L}_i)^j \right) \\ &\quad - \lambda_{i,k}\bar{L}_i(h_{i,k} + p_{i,k}) \\ &\quad \left. \times \left(- \sum_{j=0}^{S_{i,k}-m_{i,k}-2} \frac{1}{j!} \times \theta \right) + p_{i,k}\lambda_{i,k} \right] \\ &= \beta_{i,k}\lambda_{i,k} \end{aligned} \tag{10}$$

where $\theta = -\lambda_{i,k} e^{-\lambda_{i,k}\bar{L}_i} (\lambda_{i,k}\bar{L}_i)^j + e^{-\lambda_{i,k}\bar{L}_i} \times j \cdot (\lambda_{i,k}\bar{L}_i)^{j-1} \cdot \lambda_{i,k}$.

4.2. The coordination procedure

As mentioned in Section 4.1, it is an iterative procedure which solves alternatively the RDC decision problems and the CDC decision problem in each iteration. Firstly, each expected lead-time $\bar{L}_i$ is initialised with a value in the interval $[l_i, l_i + L_0]$, $i \in N$. Given the expected lead-times $\bar{L} = \{\bar{L}_i, i \in N\}$, the RDC problems (3) are solved individually and independently to obtain their tentative order-up-to levels $S_{i,k}$, $i \in N$, $k \in K$, based on which the shortage cost coefficients $\beta = \{\beta_i, i \in N\}$ can be calculated, where $\beta_i = (\beta_{i,k}, k \in K)$. Knowing these shortage cost coefficients, the CDC problem (9) is solved to obtain its tentative order-up-to levels $S_{0,k}, k \in K$. The new values of $S_{0,k}, k \in K$ will then be used to determine the new values of expected lead-times $\bar{L}$ with which the RDC problems are solved again. The updating of $\bar{L}$, the RDC problems, β, and the CDC problem is repeated until a stationary solution is found for the CDC problem and the RDC problems, if such a solution exists. A stationary solution $\{S_0, S_i, i \in N\}$ for these problems is a solution that does not change during two successive iterations. From a managerial point of view, the coordination procedure can be interpreted as a negotiation process, in which the CDC reimburses each retailer the cost induced by its late deliveries. The negotiations and successive updates of $\bar{L}$ and β continue until a stationary solution (or equivalently a Nash equilibrium) is found. The only information exchanged between each RDC i and the CDC is β_i, the shortage cost coefficients for late deliveries of the CDC. Our decomposition and coordination procedure is illustrated in Figure 2.

Next, we discuss how to solve the RDC problems and CDC problem. For the decision problem of each RDC i, since its objective function is item-additive (see Eq. (3)),

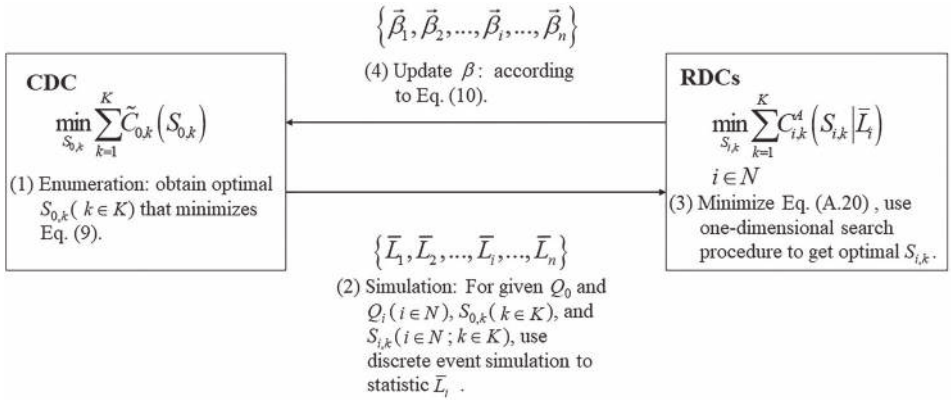

Figure 2. The procedure of the decomposition and coordination approach.

and $C_{i,k}^{A}(S_{i,k}|\bar{L}_i)$ is convex with respect to $S_{i,k}$ for a given $\bar{L}_i$, it can be solved by applying a simple one-dimensional search procedure, such as bisection method, in the interval $[S_{i,k}^{l}, S_{i,k}^{u}]$, to obtain its optimal solution $S_{i,k}$, where $S_{i,k}^{l}$ and $S_{i,k}^{u}$ are a lower bound and an upper bounds of $S_{i,k}$, respectively.

For the decision problem of the CDC, its objective function may not be convex. Since Q_0 is given, to fin the optimal solution S_0 of the problem, we first enumerate $s_0 = \sum_{k=1}^{K} S_{0,k} - Q_0$ as an integer value in the interval $[s_0^l, s_0^u]$ defining the range of s_0. For each given s_0, the objective function of the CDC becomes item-additive, i.e. $C_0(S_0) = \sum_{k=1}^{K} C_{0,k}(S_{0,k})$. Suppose that the decision variable $S_{0,k}$ takes an integer value in the interval $[S_{0,k}^{l}, S_{0,k}^{u}]$, where $S_{0,k}^{l}$ and $S_{0,k}^{u}$ are a lower bound and an upper bound of $S_{0,k}$, respectively.

Because of the item-additivity of the CDC problem, it can be formulated as a 0–1 integer program. In order to do so, we defin 0–1 integer variables:

$$x_{kj} = \begin{cases} 1, & \text{if } S_{0,k} = j, j \in [S_{0,k}^{l}, S_{0,k}^{u}] \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

The 0–1 program can be formulated as:

IP01:

$$\min TC(s_0) = \sum_{k=1}^{K} \sum_{j=S_{0,k}^{l}}^{S_{0,k}^{u}} C_{0,k}(j) x_{kj} \tag{12}$$

s.t.

$$\sum_{j=S_{0,k}^{l}}^{S_{0,k}^{u}} x_{kj} = 1, \quad \forall k \in K \tag{13}$$

$$\sum_{k=1}^{K} \sum_{j=S_{0,k}^{l}}^{S_{0,k}^{u}} j \cdot x_{kj} = s_0 + Q_0 \tag{14}$$

In the objective function (12), $TC(s_0)$ is the total cost of the CDC when s_0 is given $C_{0,k}(j)$ is the value of $C_{0,k}(S_{0,k})$ when $S_{0,k} = j$. $C_{0,k}(j)$, $j \in [S_{0,k}^{l}, S_{0,k}^{u}]$ can be obtained by discrete event simulation. Constraint (13) implies that $S_{0,k}$ can only take a single value in interval $[S_{0,k}^{l}, S_{0,k}^{u}]$ for any item $k \in K$. Constraint (14) ensures $s_0 = \sum_{k=1}^{K} S_{0,k} - Q_0$.

By enumerating integer s_0 in the interval $[s_0^l, s_0^u]$ and solving the 0–1 program for each possible s_0, we can obtain the optimal solution of the CDC problem by comparing the optimal objective values of these 0–1 programs. This solution is denoted by S_0^*.

To obtain lower bound $S_{i,k}^{l}$ and upper bound $S_{i,k}^{u}$ for all RDCs and the CDC, i.e. for any $i \in N \cup 0$, we fi st run the decomposition and coordination procedure from step (3) with the expected lead-times, $\bar{L}_i$, $i \in N$ initialised to their smallest possible values, i.e. $\bar{L}_i = l_i$ for $i \in N$. Due to the concavity of $C_{i,k}^{A}(S_{i,k}^{*}(\bar{L}_i)|\bar{L}_i)$ with respect to $\bar{L}_i$ and the fact that $S_{0,k}$ is a non-decreasing function of β_i, this procedure will end with a stationary solution, $(S_{0,k}^{u}, S_{i,k}^{l} | i \in N, k \in K)$, where $S_{0,k}^{u}$ is an upper bound of $S_{0,k}$ and $S_{i,k}^{l}$ is a lower bound of $S_{i,k}$, $i \in N$. We then rerun the procedure starting from step (3) with the expected lead-times, $\bar{L}_i$, $i \in N$, initialised to their largest possible values, i.e. $\bar{L}_i = l_i + L_0$ for $i \in N$. At this time, this procedure will end with a stationary solution $(S_{0,k}^{l}, S_{i,k}^{u} | i \in N, k \in K)$, where $S_{0,k}^{l}$ is a lower bound of $S_{0,k}$ and $S_{i,k}^{u}$ is an upper bound of $S_{i,k}$, $i \in N$. If $S_{0,k}^{l} = S_{0,k}^{u}$ for all $k \in K$, then the optimal solution of the approximate model has been found. Otherwise, we must consider a fin te number of solutions in the interval $[S_{0,k}^{l}, S_{0,k}^{u}]$ for each $k \in K$. Remember that each value of $S_{0,k}$, through the expected lead-time $\bar{L}_i$, uniquely determines the optimal order-up-to level $S_{i,k}^{*}$ of RDC i.

Steps (1)-(2)-(3)-(4) in Figure 2 form a closed-loop for each iteration of the decomposition and coordination procedure. After calculating lower bound $S_{i,k}^{l}$ and $S_{i,k}^{u}$, i

$\in N$ for all stocks as mentioned above, this procedure is run again but it now starts from step (1) with all induced shortage cost coeffici ts initialised to zero, i.e. $\beta_{i,k} = 0$, $i \in N$, $k \in K$. The iterative process of this procedure will repeat until a stationary solution is found.

Our decomposition and coordination approach is presented in pseudo-code as follows:

5. Error bound

In this section, we discuss the performance evaluation of the solution found by our decomposition and coordination approach. Firstly, define:

$S_{i,k}^{E*}(S_0)$: true optimal order-up-to level of item k at RDC i given order-up-to levels S_0 of the CDC in the exact model, where $S_0 = (S_{0,k}, k \in K)$.
$S_{i,k}^{A*}(S_0)$: optimal order-up-to level of item k at RDC i given order-up-to levels S_0 of the CDC in the approximate model.
$(S_0^{E*}, S_i^{E*}, i \in N)$ is the optimal solution of the exact model, where $S_0^{E*} = (S_{0,k}^{E*}, k \in K)$, $S_{i,k}^{E*} = (S_{i,k}^{E*}, k \in K)$, $i \in N$.
$(S_0^{A*}, S_i^{A*}, i \in N)$ is the approximate solution found by Algorithm 1 based on the approximate model, where $S_0^{A*} = (S_{0,k}^{A*}, k \in K)$, $S_{i,k}^{A*} = (S_{i,k}^{A*}, k \in K)$, $i \in N$.
$C_{i,k}^{E}(S_{i,k}^{E*}(S_0))$: minimum cost for item k at RDC i in the exact model given order-up-to levels S_0 of the CDC.
$C_{i,k}^{A}(S_{i,k}^{A*}(S_0))$: minimum cost for item k at RDC i given order-up-to levels S_0 of the CDC in the approximate model. $C_{i,k}^{A}(S_{i,k}^{A*}(S_0)) = C_{i,k}^{A}(S_{i,k}^{*}|\bar{L}_i(S_0))$.

Algorithm 1: Decomposition–Coordination Approach

Input: A predefined accuracy e for the bisection search of $S_{i,k}$ and a sufficiently long simulation time horizon T for evaluating $\bar{L}_i$ and $C_{0,k}(S_{i,k})$, $S_{i,k} \in [S_{0,k}^{l}, S_{0,k}^{u}]$.
Result: Output $(S_{0,k}^{*}, S_{i,k}^{*} | i \in N, k \in K)$
1 Determine Q_0 and Q_i, $i \in N$ based on a deterministic multi-echelon joint replenishment model (see Appendix A).
2 Set $\bar{L}_i = l_i$ for $i \in N$, and iterate the procedure in Figure 2 from step (3) until it converges to a stationary solution to obtain $(S_{0,k}^{u}, S_{i,k}^{l} | i \in N,\ k \in K)$.
3 Set $\bar{L}_i = l_i + L_0$ for $i \in N$, and iterate the procedure in Figure 2 from step (3) until it converges to a stationary solution to obtain $(S_{0,k}^{l}, S_{i,k}^{u} | i \in N, k \in K)$.
4 Initialize the induced shortage cost coefficients $\beta_{i,k} = 0$, $i \in N$, $k \in K$.
5 **While** the stop criterion is not achieved, i.e. a stationary solution is not found, **do**
6 Calculate $C_{0,k}(S_{i,k})$, $S_{i,k} \in [S_{0,k}^{l}, S_{0,k}^{u}]$ by discrete event simulation.
7 Solve the decision problem of the CDC to find its tentative optimal order-up-to levels $S_{0,k}^{*}$, $k \in K$ by solving the 0–1 programming **IP01**.
8 Calculate $\bar{L}_i(S_{0,k}^{*}, k \in K)$, $i \in N$ by discrete event simulation.
9 Solve the decision problem of each RDC to obtain its tentative optimal order-up-to levels $S_{i,k}^{*}$, $k \in K$ by the bisection search.
10 Update $\beta_{i,k}$, $i \in N$, $k \in K$ according to Eq. (10).
11 **End while**
12 Output $(S_{0,k}^{*}, S_{i,k}^{*} | i \in N, k \in K)$

For the approximate model and the exact model, following Andersson, Axsäter, and Marklund (1998) and Andersson and Marklund (2000), we have $C_{i,k}^{A}(S_{i,k}^{A*}(S_0)) \le C_{i,k}^{E}(S_{i,k}^{E*}(S_0))$ for any i, k and S_0, and $TC^{A}(S_0^{A*}, S_i^{A*}, i \in N) \le TC^{E}(S_0^{E*}, S_i^{E*}, i \in N)$.

The performance of the solution found by our approach can thus be evaluated by the ratio ε defin d as:

$$\varepsilon = \frac{TC^{E}(S_0^{A*}, S_i^{A*}, i \in N)}{TC^{E}(S_0^{E*}, S_i^{E*}, i \in N)} \tag{15}$$

If $\varepsilon = 1$, the solution is optimal, otherwise $\varepsilon > 1$. The larger ε, the worst the approximate solution. Unfortunately, ε cannot be obtained since the denominator in (15) involves the optimal solution, which is unknown. However, since $TC^{A}(S_0^{A*}, S_i^{A*}, i \in N) \le TC^{E}(S_0^{E*}, S_i^{E*}, i \in N)$, by replacing the denominator $\boldsymbol{TC^{E}(S_0^{E*}, S_i^{E*}, i \in N)}$ in (15) by $TC^{A}(S_0^{A*}, S_i^{A*}, i \in N)$, we can obtain an upper bound $\hat{\varepsilon}$ of ε, where

$$\hat{\varepsilon} = \frac{\sum_{k=1}^{K} C_{0,k}(S_{0,k}^{A*}) + \sum_{i=1}^{N}\sum_{k=1}^{K} E_{L_i(S_0^{A*})}[C_{i,k}^{A}(S_{i,k}^{A*}|L_i(S_0^{A*}))]}{TC^{A}(S_0^{A*}, S_i^{A*}, i \in N)} \tag{16}$$

In order to evaluate the upper bound of the performance ratio, we must know the probability distribution of $L_i(S_0^{A*})$, which can be obtained by discrete event simulation of the distribution system with given inventory policies and applying a discrete distribution fitting technique.

6. Numerical study

In this section, we report our numerical study for evaluating the performance of our proposed approach for optimising $(Q, \boldsymbol{S})$ policies in a distribution system and provide some managerial insights for managing distribution systems with joint replenishment and real applications of the approach.

6.1. Instance generation

According to the characteristics of the two-echelon distribution system studied, we randomly generate many instances that represent the system in different instances. Since the impact of each parameter (e.g. lead time of the CDC) on the performance of the system mainly depends on its relative size concerning the size of the corresponding other parameter(s) (e.g. lead time of each RDC), we fi the parameters of the CDC (or RDCs) while varying the parameters of RDCs (or the CDC) to generate multiple instances. The possible values for each parameter are summarised in Table 2 and explained below.

According to Table 2, the distribution system may have 2, 4, or 6 RDCs and 3, 4, or 5 items. The

Table 2. Possible values for each parameter.

Parameters	Values
N	2, 4, 6
K	3, 4, 5
L_0	1, 2, 4
l_i	2
$\lambda_{i,k}$	Randomly generated, see Table 3 for details.
A_0	$A_0 + \sum_{k=1}^{K} a_{0,k} = 0.5ct_0^2 \sum_{k=1}^{K} h_{0,k} \cdot \left(\sum_{i=1}^{N} \lambda_{i,k}\right)$, $ct_0 \in$ {U[1, 3], U[2, 6]}.
$a_{0,k}$	$\frac{a_{0,k}}{A_0} \in$ U[0.1, 0.3]
A_i	$A_i + \sum_{k=1}^{K} a_{i,k} = 0.5ct_i^2 \sum_{k=1}^{K} h_{i,k}\lambda_{i,k}$, $ct_i \in$ {U[1, 3], U[2, 6]}
$a_{i,k}$	$\frac{a_{i,k}}{A_i} \in$ U[0.1, 0.3]
$h_{0,k}$	$h_{0,k} \in$ U[1, 10]
$h_{i,k}$	$\frac{h_{i,k}}{h_{0,k}} \in$ U[1, 2]
$p_{i,k}$	$\frac{p_{i,k}}{h_{i,k}} \in$ U[10, 40]

constant transportation time l_i between the CDC and each RDC i is set to 2 for all RDCs, the constant replenishment lead-time L_0 of the CDC is set to three possible values 1, 2, 4. The remaining parameters are randomly generated. Major ordering cost A_0 and minor ordering costs $a_{0,k}$ for the CDC, are set according to $A_0 + \sum_{k=1}^{K} a_{0,k} = 0.5ct_0^2 \sum_{k=1}^{K} h_{0,k} \cdot \left(\sum_{i=1}^{N} \lambda_{i,k}\right)$, where $\lambda_{i,k}$ is the mean demand rate for item k at RDC i and ct_0 can be interpreted as the expected order cycle time under the economic order quantity model, here $ct_0 \in$ {U[1, 3],U[2, 6]} (Corresponding to $A_0 \in \{A_0^S, A_0^L\}$). The relationship between $a_{0,k}$ and A_0 follows $\frac{a_{0,k}}{A_0} \in$ U[0.1, 0.3], according to the guidelines of Pantumsinchai (1992). Similarly, major ordering cost A_i and minor ordering costs $a_{i,k}$ for each RDC i, are set according to $A_i + \sum_{k=1}^{K} a_{i,k} = 0.5ct_i^2 \sum_{k=1}^{K} h_{i,k}\lambda_{i,k}$, with its expected order cycle time set as $ct_i \in$ {U[1, 3], U[2, 6]} (Corresponding to $A_i \in \{A_i^S, A_i^L\}$). There is a proportional relationship between $a_{i,k}$ and A_i for RDC i expressed as $\frac{a_{i,k}}{A_i} \in$ U[0.1, 0.3]. Parameter $h_{0,k}$ is set in the same way as in most literature, the holding cost per unit of item per unit of time for each item k at the CDC is randomly generated by $h_{0,k} \in$ U[1, 10]. The shortage cost per unit of item per unit of time for item k at RDC i is set as $\frac{p_{i,k}}{h_{i,k}} \in$ U[10, 40], which correspond to expected service level $\alpha \in$ U[90%, 97.5%].

The parameters $\lambda_{i,k}$ are randomly generated in the following way: we first randomly generate the demand rates for all items in RDC1, and then use them as references to generate the demand rates of all items in RDC i other than RDC 1 by $\lambda_{i,k} = r_i^d \cdot \lambda_{1,k}$, where the coefficient r_i^d is randomly generated from the uniform distribution U[1, i] (Table 3).

Table 3. Random generation of parameter $\lambda_{i,k}$.

RDC i	Demand Multiplier r_i^d	Demand Rate $\lambda_{i,k}$ Item 1	Item 2	Item 3	Item 4	Item 5
RDC1	$r_1^d = 1$	U[1, 5]	U[1, 10]	U[1, 15]	U[1, 20]	U[1, 25]
RDC2	$r_2^d \in$ U[1, 2]	$\lambda_{i,k} = r_i^d \cdot \lambda_{1,k}$				
RDC3	$r_3^d \in$ U[1, 3]					
RDC4	$r_4^d \in$ U[1, 4]					
RDC5	$r_5^d \in$ U[1, 5]					
RDC6	$r_5^d \in$ U[1, 6]					

Table 4. Parameters of nine sets of instances.

Instance set	N	K	L_0	A_0, A_i
IS1 (set of referential instances)	4	4	2	A_0^S, A_i^S
IS2	**2**	4	2	A_0^S, A_i^S
IS3	**6**	4	2	A_0^S, A_i^S
IS4	4	**3**	2	A_0^S, A_i^S
IS5	4	**5**	2	A_0^S, A_i^S
IS6	4	4	**1**	A_0^S, A_i^S
IS7	4	4	**4**	A_0^S, A_i^S
IS8	4	4	2	A_0^S, A_i^L
IS9	4	4	2	A_0^L, A_i^S

6.2. Parameter setting

We generated 9 sets of instances or scenarios, each containing 20, and 180 instances were generated. As mentioned in the previous subsection, $l_i = 2$ for all instances/scenarios. Among these sets, the first one, referred to as IS1 for short, is a set of instances for evaluating the performance of the proposed approach and referential for the sensitivity analysis of some key parameters of the distribution system. For this set of instances, the parameters are set as: $n = 4$, $K = 4$, $L_0 = 2$, $A_0 = A_0^S, A_i = A_i^S$. The parameters $\lambda_{i,k}$, A_0, $a_{0,k}$, A_i, $a_{i,k}$, $h_{0,k}$, $h_{i,k}$, and $p_{i,k}$ were randomly generated according to subsection 6.1. For IS1, we randomly generated 20 instances. The other 8 sets of instances were generated by varying the value of one parameter concerning the parameter values of IS1. For example, the instance set IS2 was generated from IS1 by changing N from 4 to 2, and the instance set IS7 was generated from IS1 by changing L_0 from 2 to 4. The values of the three parameters for each instance set are given in Table 4, where the number in bold type in each line indicates the parameter whose value is changed from that in the referential instance set IS1 to that in the corresponding set.

6.3. Performance evaluation

Our proposed algorithm was implemented in C/C++ with Microsoft Visual Studio (Version 2022). All numerical experiments were carried out on a PC operating in Windows 10 with CPU i7-8650U and 16 GB RAM. Firstly, we tested the algorithm on the instance set IS1 to

evaluate its solution quality and computation time performance. For each distribution system instance in this set, we first set the value of Q for each (Q, $\boldsymbol{S}$) policy according to Appendix A, we then applied the algorithm to optimise the values of $\boldsymbol{S}$. The predefined accuracy e for the bisection method used in the algorithm is set to 0.5 and the simulation horizon T is set to 10,000. After testing several instances, we found this simulation horizon was sufficiently long to obtain highly accurate estimation of expected lead times $\bar{L}_i$ and cost functions $C_{0,k}(.)$. To demonstrate the advantage of coordination between the inventory decision of CDC and those of the RDCs realised by our algorithm, we also report the total cost of the inventory policies found by the algorithm at its first iteration with all induced shortage cost coefficients β set to zero. Note that solving the CDC problem and the RDC problems with β set to zero is similar to what done in the METRIC approach proposed by Sherbrooke (1968). The results of the 20 instances of IS1 are presented in Table 5.

In this table, the first column indicates the serial number of each instance; the 2nd to 6th columns give the parameter values of the (Q, $\boldsymbol{S}$) policies obtained by our algorithm for the distribution system of each instance; the 7th column TC^A is the approximate expected total cost per time unit of the system with the (Q, $\boldsymbol{S}$) policies, obtained based on the approximate model (2) which replaces the stochastic lead time of each RDC by its mean; the 8th column presents the error bound $\hat{\varepsilon}$ given by Equation (16); the 9th column TC^{sim} is the total cost per time unit of the system obtained by simulation; the 10th column is the CPU time of the algorithm; the 11th column TC^0 is the total cost per time unit of the system obtained by simulation with the (Q, $\boldsymbol{S}$) policies found at the first iteration of our algorithm, the (Q, $\boldsymbol{S}$) policies can be considered being those obtained by the METRIC approach; the last column gives the value of TC^0/TC^A; the last row of this table presents the mean values of TC^A, $\hat{\varepsilon}$, TC^{sim}, CPU time, TC^0 and TC^0/TC^A for 20 instances in IS1. Note that for both (Q, $\boldsymbol{S}$) policies obtained by our algorithm and the METRIC approach, the total cost of the distribution system is obtained by simulation, this is because it is very difficult to analytically calculate the expected total cost per time unit of the distribution system given its inventory policies. By simulating the system for a sufficiently long time, its average total cost per time unit, i.e. TC^{sim}, is an effective upper bound of its expected total cost, whereas TC^A is a lower bound of the expected total cost.

From Table 5, we can see our algorithm's CPU time for each instance ranges from 1452 to 2144 s. This computation time is reasonable since the inventory optimisation problem considered is a tactical decision problem. The range of error bound $\hat{\varepsilon}$ is from 1.03–1.17

Table 5. Results of referential instances (Instance Set 1).

Instance no.	$\left(\frac{Q_0}{S_0}\right)$	$\left(\frac{Q_1}{S_1}\right)$	$\left(\frac{Q_2}{S_2}\right)$	$\left(\frac{Q_3}{S_3}\right)$	$\left(\frac{Q_4}{S_4}\right)$	TC^A	$\hat{\varepsilon}$	TC^{sim}	CPU Time(s)	TC^0	TC^0/TC^A
IS1-1	310, (42,60,190,49)	32, (13,19,53,16)	72, (21,30,93,26)	74, (28,39,118,36)	53, (20,29,85,24)	863.3	1.06	915.1	1452	1079.13	1.25
IS1-2	255, (76,68,67,110)	40, (31,28,30,40)	78, (45,40,42,59)	61, (44,40,44,57)	103, (74,62,69,94)	1234.78	1.07	1321.21	1921	1704	1.38
IS1-3	419, (34,134,201,166)	80, (14,47,67,63)	150, (24,82,114,110)	159, (25,91,126,122)	159, (32,111,156,146)	2764.6	1.06	2930.48	1798	3870.44	1.4
IS1-4	343, (39,57,33,276)	52, (16,21,13,106)	99, (24,32,18,153)	185, (42,56,32,279)	153, (34,46,27,236)	702.3	1.08	758.48	1836	849.78	1.21
IS1-5	367, (49,46,42,322)	49, (18,15,15,97)	79, (25,23,23,153)	140, (35,32,29,220)	147, (45,43,40,277)	1776.75	1.08	1918.89	1566	2220.94	1.25
IS1-6	328, (63,32,270,22)	62, (23,14,97,10)	108, (38,22,170,18)	95, (43,26,182,18)	120, (51,30,225,24)	1363.15	1.04	1417.68	1698	1853.88	1.36
IS1-7	487, (66,43,297,144)	71, (21,15,91,43)	95, (27,18,123,57)	100, (28,19,128,56)	220, (57,37,262,125)	1778.37	1.08	1920.64	1872	2489.72	1.4
IS1-8	529, (97,80,246,226)	96, (32,32,80,80)	126, (50,51,123,128)	182, (69,67,179,185)	239, (73,74,188,192)	2487.33	1.09	2711.19	2107	3158.91	1.27
IS1-9	347, (56,112,37,206)	35, (17,32,11,53)	82, (29,53,18,88)	84, (30,57,19,93)	128, (44,83,27,139)	2623.28	1.1	3016.77	2134	3646.36	1.39
IS1-10	513, (21,158,102,381)	101, (9,61,37,125)	179, (17,100,68,221)	71, (9,53,35,109)	255, (26,176,118,376)	2389.2	1.1	2628.12	2098	3105.96	1.3
IS1-11	813, (79,231,156,505)	70, (19,48,32,106)	108, (28,70,52,162)	205, (53,132,86,295)	159, (42,99,69,232)	2987.18	1.06	3166.41	1520	4271.67	1.43
IS1-12	459, (31,172,126,269)	57, (11,50,36,74)	121, (18,78,57,120)	123, (23,98,75,153)	294, (39,179,134,281)	2460	1.07	2632.2	2002	3222.6	1.31
IS1-13	483, (24,218,168,187)	55, (9,57,45,58)	107, (17,105,78,106)	129, (18,126,96,124)	290, (32,220,163,214)	2532.63	1.17	2963.18	1569	3317.75	1.31
IS1-14	199, (23,52,68,93)	51, (13,26,38,50)	61, (20,38,55,75)	91, (22,44,64,85)	65, (15,31,47,63)	1122.29	1.07	1200.85	2144	1616.1	1.44
IS1-15	166, (51,31,36,91)	30, (18,14,15,32)	39, (27,22,22,50)	47, (34,24,26,61)	91, (48,35,40,90)	885.22	1.03	911.78	1648	1248.16	1.41
IS1-16	756, (43,190,214,426)	112, (13,61,66,116)	129, (20,88,93,166)	113, (17,73,82,149)	186, (27,122,127,241)	1361.52	1.1	1497.67	2011	1783.59	1.31
IS1-17	364, (33,47,207,157)	61, (16,19,85,62)	76, (21,25,100,75)	158, (39,49,209,149)	105, (20,28,112,83)	1859.67	1.05	1952.65	2131	2417.57	1.3
IS1-18	434, (65,42,182,237)	64, (26,15,59,86)	120, (37,22,85,127)	85, (33,21,79,114)	309, (93,53,211,311)	2955.96	1.07	3162.88	1637	3694.95	1.25
IS1-19	760, (102,116,315,378)	58, (23,25,56,71)	119, (34,42,90,106)	169, (60,73,153,190)	201, (72,83,179,219)	3548.3	1.06	3761.2	1855	4328.93	1.22
IS1-20	293, (59,41,21,225)	39, (19,17,9,75)	87, (33,27,14,127)	80, (36,31,16,147)	172, (59,48,26,241)	1516.63	1.16	1759.29	1825	1986.79	1.31
Mean						1960.62	1.08	2127.334	1841.2	2593.36	1.33

Table 6. Variation of error bound and CPU time as each parameter changes in the referential instances.

	Error Bound $\hat{\varepsilon}$				CPU Time(s)			
Instance set	Mean	Std Dev	Min	Max	Mean	Std Dev	Min	Max
IS1 (set of referential instances)	1.08	0.0346	1.03	1.17	1841.2	225.5234	1452	2144
N ↓	1.073	0.0298	1.03	1.16	919.85	112.195	740	1077
N ↑	1.068	0.0442	1.02	1.17	2821.6	268.9875	2223	3174
K ↓	1.068	0.0323	1.02	1.15	1294.75	118.68	1108	1531
K ↑	1.076	0.0365	1.01	1.16	2255.85	271.4587	1887	2662
L_0 ↓	1.063	0.0261	1.03	1.14	1820.55	186.2456	1449	2158
L_0 ↑	1.069	0.037	1.02	1.16	1762.7	216.0843	1462	2155
A_0^S, A_i^L	1.083	0.0315	1.03	1.16	1787.4	212.3362	1467	2125
A_0^L, A_i^S	1.062	0.0352	1.02	1.15	1868.35	252.7629	1452	2356

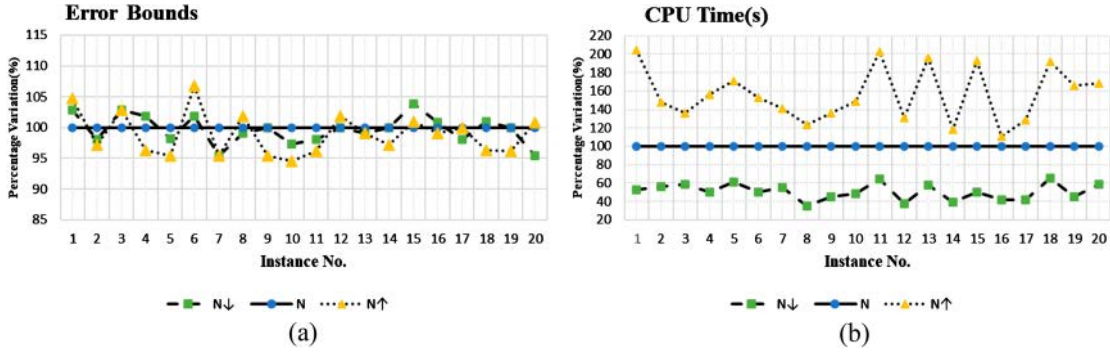

Figure 3. The percentage variation of error bound and CUP time with respect to *N*.

with average 1.08. This implies our approach can obtain high-quality near-optimal inventory policies. Moreover, TC^0/TC^A is signifi antly larger than $\hat{\varepsilon}$ for all instances, this implies the coordination between the CDC and the RDCs realised by our approach can significantly improve the inventory policies found. In addition to this instance set, we tested all other instance sets in Table 4 with totally 180 instances tested. In summary, our coordination procedure performs well with an average error bound of about 7%. Refer to Appendix C for detailed computation results of all instance sets.

6.4. Sensitivity analysis

We perform a sensitivity analysis on parameters N, K, L_0, and A of the distribution system. The performance indexes considered for this analysis are error bound $\hat{\varepsilon}$ and CPU time. We provide the mean value, standard deviation, minimum and maximum value of the error bound and CPU time of 20 instances in each instance set. The results are summarised in Table 6.

In Table 6, the firs column indicates the variation of each parameter with respect to its value in the referential set IS1, with decrease symbol ↓ and increase symbol ↑. The 2nd to 9th columns give the mean value, standard deviation, minimum and maximum value of the error bound and the CPU time respectively for the 20 instances in each set.

The detailed results of the sensitivity analysis are illustrated in the following figures.

From Table 6 and Figures 3–6, we can observe: (1) The coordination procedure is quite robust regarding solution quality, there is no signific nt impact of the change of parameters N, K, L_0, and A on its error bound. (2) Parameters N and K signific ntly impact the CPU time of the procedure. The CPU time increases as the increase of the number of RDCs and/or the number of items. According to our numerical experiments, the computation time of the approach increases linearly rather than exponentially as the number of items increases, so our proposed approach has good scalability in its real implementation.

6.5. Managerial insights and real implementation of the proposed approach

Optimising inventory policies in distribution systems poses more challenges than serial or assembly systems. The complexity of this optimisation further increases when multiple items are jointly replenished in a distribution system. As a result, many enterprises opt to optimise an inventory policy individually for each stocking location or each item without considering coordination among related stocks/items, or they adopt a sequential optimisation method like the METRIC approach.

Our study demonstrates that the coordination among the inventory decisions of multiple stocking locations and the coordination among replenishments of multiple items can significantly improve the performance of

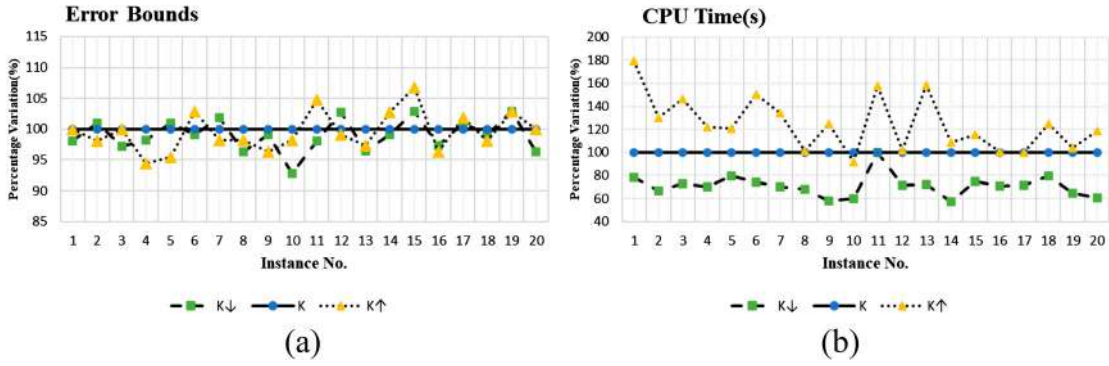

Figure 4. The percentage variation of error bound and CPU time with respect to K.

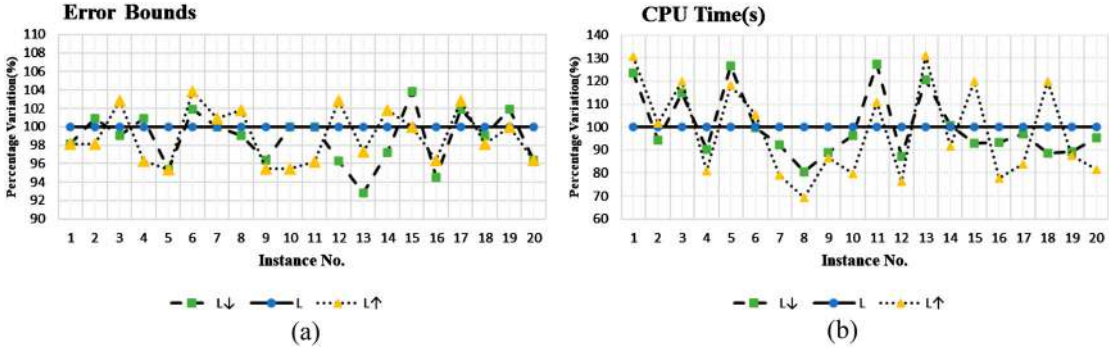

Figure 5. The percentage variation of error bound and CPU time with respect to L_0.

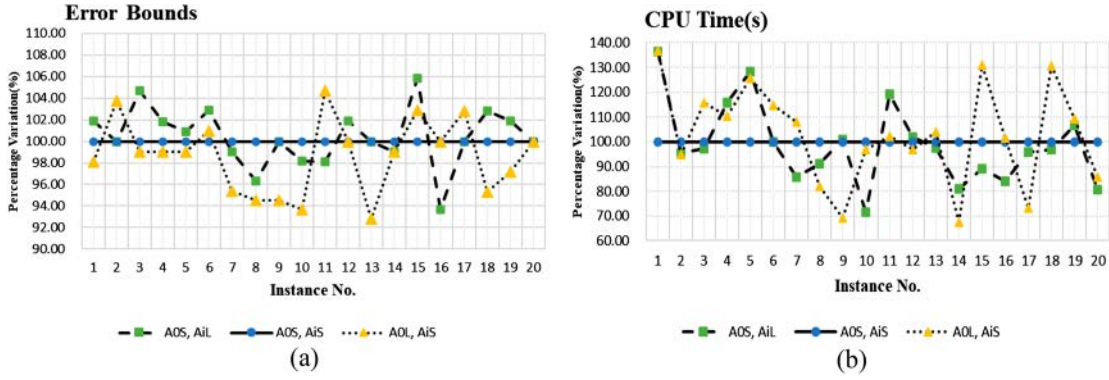

Figure 6. The percentage variation of error bound and CPU time with respect to A_0, A_i.

a distribution system. Considering the interdependencies among multiple stocking locations and the aggregate demand of multiple items in each stocking location, we can obtain inventory policies much better than those obtained by the METRIC approach with a significant cost reduction.

Our proposed decomposition and coordination approach is effective, efficient, very general and robust, making it promising for real implementation. With some adjustments to accommodate specific demand patterns, this approach can be extended to distribution systems with various demands such as compound Poisson demands or normally distributed demands.

7. Conclusion

In this paper, we have studied the optimisation of $(Q, \boldsymbol{S})$ policies for a two-echelon distribution system with joint inventory replenishment of multiple items. After deriving analytically the cost functions of the CDC and all RDCs in the system in an approximate model, we have proposed an iterative decomposition and coordination approach for solving the model. The results of our numerical study show that this approach is effective and computationally efficient, it can find high-quality near-optimal $(Q, \boldsymbol{S})$ policies of the system in a reasonable computation time, and the coordination between inventory decisions of the CDC and those of the RDCs realised by this approach can

significantly improve the performance of the distribution system.

Our proposed approach in its current version is limited to two-echelon distribution systems with Poisson demands, (Q, $\boldsymbol{S}$) policy for joint replenishment of each stock, and the FCFS rule for inventory allocation of each upstream stock. Our future work will extend this approach to more general distribution systems with compound Poisson demands or normally distributed demands, other policies for joint replenishment of each stock and inventory allocation of each upstream stock.

Disclosure statement

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Data availability statement

The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

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Appendices

Appendix A. Determination of *Q* for each (*Q*, *S*) policy in the distribution system

In this appendix, we propose a heuristic approach to determine *Q* for each (*Q*, ***S***) policy in the distribution system studied based on a deterministic multi-echelon joint replenishment model. In this model, the stochastic demand of each item at each RDC is replaced by its mean value. This model assumes a common replenishment cycle time for all items at each stock and in each replenishment cycle of the CDC, one or multiple replenishments is (are) performed by each RDC, i.e. the cycle time of the CDC is a multiple of the cycle time of each RDC.

Let us define:

T_0: the replenishment cycle time of the CDC.
T_i: the replenishment cycle time of RDC *i*.

We impose the following constraints in the model:

$$T_0 = r_i T_i \tag{A.1}$$

$$r_i \geq 0 \text{ and integer }, i = 1, \ldots, \mathrm{N} \tag{A.2}$$

Constraint (A.2) is added since in real distribution systems especially in E-commerce distribution systems, each downstream RDC usually replenishes its inventory more frequently than the upstream CDC.

If the replenishment quantity of each item *k* in each joint replenishment of RDC *i* is $Q_{i,k}$, and the replenishment quantity of each item *k* in each joint replenishment of the CDC is $Q_{0,k}$, then they have the following relationship:

$$Q_{0,k} = \sum_{i=1}^{\mathrm{N}} r_i Q_{i,k} \tag{A.3}$$

Let $Q_0 = \sum_{k=1}^{\mathrm{K}} Q_{0,k}$ for the CDC and $Q_i = \sum_{k=1}^{\mathrm{K}} Q_{i,k}$ for each RDC *i*.

According to Eq. (A.1) and (A.3), $Q_{i,k} = \lambda_{i,k} T_i$, and $Q_{0,k} = \sum_{i=1}^{\mathrm{N}} \lambda_{i,k} T_0$, we can get:

$$Q_{i,k} = Q_{0,k} \frac{1}{r_i} \frac{\lambda_{i,k}}{\sum_{i=1}^{\mathrm{N}} \lambda_{i,k}} \tag{A.4}$$

The total ordering and holding cost per time unit of RDC *i* for each item *k* and all items in the long-term operation can be written as:

$$C_{i,k}(Q_{i,k}) = \frac{\lambda_{i,k}}{Q_{i,k}} a_{i,k} + \frac{Q_{i,k}}{2} h_{i,k} \tag{A.5}$$

$$C_i(Q_i) = \frac{\lambda_{i,k}}{Q_{i,k}} \left(A_i + \sum_{k=1}^{\mathrm{K}} a_{i,k} \right) + \sum_{k=1}^{\mathrm{K}} \frac{Q_{i,k}}{2} h_{i,k} \tag{A.6}$$

The total ordering and holding cost per time unit of the CDC for each item *k* and all items in the long-term operation can be written as:

$$C_{0,k}(Q_{0,k}) = \frac{1}{T_0} \left[a_{0,k} + \sum_{i=1}^{\mathrm{N}} \frac{(r_i - 1) r_i}{2} Q_{i,k} T_i h_{0,k} \right] \tag{A.7}$$

$$C_0(Q_0) = \frac{1}{T_0} \left[A_0 + \sum_{k=1}^{\mathrm{K}} a_{0,k} + \sum_{i=1}^{\mathrm{N}} \sum_{k=1}^{\mathrm{K}} \frac{(r_i - 1) r_i}{2} Q_{i,k} T_i h_{0,k} \right] \tag{A.8}$$

The total ordering and holding cost per time unit of the system to minimise is the sum of those of the CDC and all RDCs, which can be written as:

$$C(Q_0, Q_1, \ldots, Q_{\mathrm{N}}) = C_0(Q_0) + \sum_{i=1}^{\mathrm{N}} C_i(Q_i)$$

$$= \frac{1}{T_0}\left[A_0 + \sum_{k=1}^{K} a_{0,k} + \sum_{i=1}^{N}\sum_{k=1}^{K} \frac{(r_i-1)r_i}{2} Q_{i,k} T_i h_{0,k}\right] + \sum_{i=1}^{N}\left[\frac{\lambda_{i,k}}{Q_{i,k}}\left(A_i + \sum_{k=1}^{K} a_{i,k}\right) + \sum_{k=1}^{K}\frac{Q_{i,k}}{2} h_{i,k}\right] \quad (A.9)$$

Since each Q_i ($i \in N$) is a function of r_i and Q_0, according to Eq. (A.1), Eq. (A.3) and Eq. (A.4), Eq. (A.9) can be written as:

$$\begin{aligned} &C(r_1, r_2, \ldots, r_N, Q_0) \\ &= \frac{1}{Q_{0,k}} \sum_{i=1}^{N} \lambda_{i,k}\left[A_0 + \sum_{k=1}^{K} a_{0,k} + r_i(A_i + a_{i,k})\right] \\ &\quad + \sum_{i=1}^{N}\sum_{k=1}^{K} \frac{Q_{0,k}}{2} \cdot \frac{1}{r_i} \cdot \frac{\lambda_{i,k}}{\sum_{i=1}^{N}\lambda_{i,k}}[(r_i-1)h_{0,k} + h_{i,k}] \\ &= \frac{1}{Q_0} \sum_{i=1}^{N}\sum_{k=1}^{K} \lambda_{i,k}\left[A_0 + \sum_{k=1}^{K} a_{0,k} + r_i(A_i + a_{i,k})\right] \\ &\quad + \frac{Q_0}{2}\left[\sum_{k=1}^{K} h'_{0,k} + \sum_{i=1}^{N}\sum_{k=1}^{K}\left(\frac{1}{r_i}\frac{\lambda_{i,k}}{\sum_{i=1}^{N}\lambda_{i,k}} h'_{i,k}\right)\right] \end{aligned} \quad (A.10)$$

In Eq. (A.10), $h'_{i,k}$ is an increment of unit holding cost. In the two-echelon distribution inventory system considered, $h'_{i,k} = h_{i,k} - h_{0,k}$, $h'_{0,k} = h_{0,k}$. If we denote $\sum_{i=1}^{N}\sum_{k=1}^{K}\lambda_{i,k}$ in Eq. (A.10) by λ, $A_0 + \sum_{k=1}^{N} a_{0,k} + r_i(A_i + a_{i,k})$ by A, and $\sum_{k=1}^{K} h'_{0,k} + \sum_{i=1}^{N}\sum_{k=1}^{K}\left(\frac{1}{r_i}\frac{\lambda_{i,k}}{\sum_{i=1}^{N}\lambda_{i,k}} h'_{i,k}\right)$ by h, the right side of Eq. (A.9) can be rewritten as $\frac{\lambda}{Q_0}A + \frac{Q_0}{2}h$, which is completely consistent in form with the Economic Order Quantity (EOQ) Model if we regard λ, A, h as the (mean) demand rate, fixed ordering cost, and unit holding cost, respectively. For this EOQ model, the optimal order quantity is $Q^* = \sqrt{\frac{2\lambda A}{h}}$, and the corresponding minimum total cost per time unit is:

$$C(Q^*) = \sqrt{2\lambda A h} \quad (A.11)$$

Therefore, the order quantity Q_0 that minimises the total cost in Eq. (A.10) can be formulated as:

$$Q_0^*(r_1, r_2, \ldots, r_N) = \sqrt{\frac{2\left(\sum_{i=1}^{N}\sum_{k=1}^{K}\lambda_{i,k}\right)\left(A_0 + \sum_{k=1}^{K} a_{0,k} + \sum_{i=1}^{N}\sum_{k=1}^{K} r_i(A_i + a_{i,k})\right)}{\left(\sum_{k=1}^{K} h'_{0,k} + \sum_{i=1}^{N}\sum_{k=1}^{K}\left(\frac{1}{r_i}\frac{\lambda_{i,k}}{\sum_{i=1}^{N}\lambda_{i,k}} h'_{i,k}\right)\right)}} \quad (A.12)$$

and the corresponding minimum total cost per time unit can be formulated as:

$$C(r_1, r_2, \ldots, r_N) = \sqrt{2\left(\sum_{i=1}^{N}\sum_{k=1}^{K}\lambda_{i,k}\right)\left(A_0 + \sum_{k=1}^{K} a_{0,k} + \sum_{i=1}^{N}\sum_{k=1}^{K} r_i(A_i + a_{i,k})\right)\left(\sum_{k=1}^{K} h'_{0,k} + \sum_{i=1}^{N}\sum_{k=1}^{K}\left(\frac{1}{r_i}\frac{\lambda_{i,k}}{\sum_{i=1}^{N}\lambda_{i,k}} h'_{i,k}\right)\right)} \quad (A.13)$$

Eq. (A.13) is a function of $r_1, r_2, \ldots, r_N$. If we want to minimise it, we only need to minimise the part within the root sign. Therefore, minimising Eq. (A.13) is equivalent to minimising the following function:

$$\begin{aligned} &F(r_1, r_2, \ldots, r_N) \\ &= \left(A_0 + \sum_{k=1}^{K} a_{0,k} + \sum_{i=1}^{N}\sum_{k=1}^{K} r_i(A_i + a_{i,k})\right) \\ &\quad \times \left(\sum_{k=1}^{K} h'_{0,k} + \sum_{i=1}^{N}\sum_{k=1}^{K}\left(\frac{1}{r_i}\frac{\lambda_{i,k}}{\sum_{i=1}^{N}\lambda_{i,k}} h'_{i,k}\right)\right) \end{aligned} \quad (A.14)$$

Calculating the partial derivatives of function F with respect to $r_1, r_2, \ldots, r_N$ respectively and making them equal to 0, we can obtain:

$$\frac{\partial}{\partial r_j} F(r_1, \ldots, r_j, \ldots, r_N) = 0, \quad j = 1, 2, \ldots, N \quad (A.15)$$

Each equation in (A.15) can be written as:

$$\begin{aligned} &\frac{\partial}{\partial r_j} F(r_1, r_2, \ldots, r_N) \\ &= \left(A_j + \sum_{k=1}^{K} a_{j,k}\right)\left(\sum_{k=1}^{K} h'_{0,k}\right) \\ &\quad + \left(A_j + \sum_{k=1}^{K} a_{j,k}\right)\left(\sum_{\substack{i=1 \\ (i\neq j)}}^{N}\sum_{k=1}^{K} \frac{\lambda_{i,k} h'_{i,k}}{r_i}\left(\frac{\lambda_{i,k}}{\sum_{i=1}^{N}\lambda_{i,k}}\right)\right) \\ &\quad - \frac{1}{r_j^2}\left[A_0 + \sum_{\substack{i=1 \\ (i\neq j)}}^{N} r_i\left(A_i + \sum_{k=1}^{K} a_{i,k}\right)\right] \\ &\quad \times \sum_{k=1}^{K}\left[h'_{i,k}\left(\frac{\lambda_{j,k}}{\sum_{i=1}^{N}\lambda_{i,k}}\right)\right] = 0, \; j = 1, 2, \ldots, N \end{aligned}$$

Theoretically, the values of $r_1, r_2, \ldots, r_N$ can be obtained by solving the system of equations (A.15), but it is difficult to obtain analytical expressions of $r_1, r_2, \ldots, r_N$. However, the values of $r_1, r_2, \ldots, r_N$ can be obtained by numerical calculation. Since $r_1, r_2, \ldots, r_N$ must be integers, if the solution $(r_1, r_2, \ldots, r_N)$ of the system of equations (A.15) is integer, it is feasible. Otherwise, the solution is fractional. In the latter case, the following procedure is used to find a feasible integer solution.

Step 1. For each fractional solution $(r_1, r_2, \ldots, r_N)$ obtained by solving the system of equations (A.15), round each r_j down and up to obtain two integers denoted by $\lfloor r_j \rfloor$ and $\lceil r_j \rceil$, with $\lfloor r_j \rfloor \le r_j \le \lceil r_j \rceil$.

Step 2. For each possible integer solution $r \in r \in \mathrm{R}\{\lfloor r_1 \rfloor, \lceil r_1 \rceil\} \times \{\lfloor r_2 \rfloor, \lceil r_2 \rceil\} \times \cdots \times \{\lfloor r_N \rfloor, \lceil r_N \rceil\}$, use Eq. (A.12) to calculate $Q_0^*(r)$ and the corresponding total cost $C(r, Q_0^*(r))$. Note that at most 2^N possible integer solutions need to be considered in this step.

Step 3. The required feasible integer solution $r^* = r_1^*, r_2^*, \ldots, r_N^*$ is taken as $r^* = \arg\min_{r \in \mathrm{R}} C(r, Q_0^*(r))$

After calculating $r_1^*, r_2^*, \ldots, r_N^*$, we can obtain the optimal replenishment quantities of the CDC and all RDCs: $Q_0^*, Q_1^*, Q_2^*, \ldots, Q_N^*$, with the corresponding total cost per time unit of the deterministic distribution system $C(r_1^*, r_2^*, \ldots, r_N^*, Q_0^*)$.

Appendix B. Cost functions of RDC and CDC

In this appendix, we will derive the cost functions of the RDCs and the CDC. Since the ordering cost of each stock is not affected by its order-up-to level for each item as soon as its order quantity is given. Therefore, we do not consider ordering costs in the cost functions.

The RDC cost functions

We derive the cost function of each RDC in the approximate model based on the results of Cachon (2001) on the cost evaluation of a retailer with multiple Poisson demanded products jointly replenished and controlled by a (*Q*, *S*) policy. This cost evaluation is done by relating the arrival time of each unit at the retailer to the time the unit is demanded.

In the approximate model, the lead time of each RDC i is deterministic given by its expected lead time $\bar{L}_i(S_0)$, where $S_0 = (S_{0,k}, k \in K)$ and $S_{0,k}$ is the order-up-to level of item k at RDC i. Let $\bar{L}_i = \bar{L}_i(S_0)$. For RDC i, Q_i units are ordered each time when it places an order to the CDC, and Q_i units of customer demand belonging to multiple (K) items occur between two consecutive orders. If as a part of an order placed by RDC i, an unit of customer demand of item k occurs at time $\tau_{i,k}$ and its corresponding order is shipped by the CDC to RDC i at the same time, then this unit will arrive at RDC i at time $\tau_{i,k} + \bar{L}_i$ and will be used to satisfy the $S_{i,k}$-th unit demand of item k to occur at RDC i after time $\tau_{i,k}$. In this case, the expected holding and backorder cost of RDC i incurred by this unit is given by $\hat{g}_{i,k}(S_{i,k}, \bar{L}_i)$, where:

$$\hat{g}_{i,k}(y,t) = p_{i,k}\int_0^t \gamma_{i,k}(y,x)(t-x)dx + h_{i,k}\int_t^\infty \gamma_{i,k}(y,x)(x-t)dx \tag{A.16}$$

where $\gamma_{i,k}(y,t)$ is the probability density function of the Erlang $(\lambda_{i,k}, y)$ distribution.

According to Cachon (2001), (A.16) can be further written as:

$$\hat{g}_{i,k}(y,t) = \frac{1}{\lambda_{i,k}}[y(h_{i,k}+p_{i,k})F_{i,k}(y,t) - \lambda_{i,k}t(h_{i,k}+p_{i,k})F_{i,k}(y-1,t) + p_{i,k}(\lambda_{i,k}t - y)] \tag{A.17}$$

where $F_{i,k}$ is the cumulative distribution function of the Erlang $(\lambda_{i,k}, y)$ distribution.

For an unit of customer demand of item k occurred at RDC i at time $\tau_{i,k}$, if its corresponding order i is shipped by the CDC to RDC i later at time $\tau_{i,k} + t$ $(t > 0)$ and $m_{i,k}$ units of customer demand of item k occur at RDC i in the time interval $(\tau_{i,k}, \tau_{i,k} + t]$, then it will arrive at RDC i at time $\tau_{i,k} + t + \bar{L}_i$ and will be used to satisfy the $(S_{i,k} - m_{i,k})$-th unit demand of item k to occur at RDC i after time $\tau_{i,k} + t$. In this case, the expected holding and backorder cost of RDC i incurred by this unit is given by $\hat{g}_{i,k}(S_{i,k} - m_{i,k}, \bar{L}_i)$.

Let u_i denote the number of unit demands occurred at RDC i in the interval $(\tau_{i,k}, \tau_{i,k} + t]$, $u_i \in [0, Q_i - 1]$. The u_i units belong to multiple items. According to Axsäter (1993), u_i is a uniformly distributed random variable define on the interval $[0, Q_i - 1]$. For any given u_i and any $k \in K$, let $Z_{i,k}(u_i)$ denote the number of units of item k occurred at RDC i in the interval, $Z_{i,k}(u_i)$ is a random variable. Since the demands of all items are independent, $Z_{i,k}(u_i)$ is subject to a binomial distribution with u_i as the number of draws and $\delta_{i,k} = \lambda_{i,k}/\Lambda_i$ as the probability of success. That is,

$$\begin{aligned}\Pr(Z_i(u_i) = y) &= \binom{u_i}{y}(\delta_{i,k})^y(1-\delta_{i,k})^{u_i - y} \\ &= \frac{u_i!}{y!(u_i-y)!}\left(\frac{\lambda_{i,k}}{\Lambda_i}\right)^y\left(1-\frac{\lambda_{i,k}}{\Lambda_i}\right)^{u_i-y}\end{aligned}$$

Thus, the expected holding and backorder cost per unit of item k at RDC i can be formulated as:

$$\frac{1}{Q_i}\sum_{u_i=0}^{Q_i-1}\sum_{m_{i,k}=0}^{u_i}\Pr(Z_i(u_i) = m_{i,k})\hat{g}_{i,k}(S_{i,k} - m_{i,k}, \bar{L}_i)$$

Then, by replacing $\bar{L}_i$ with $\bar{L}_i(S_0)$, the expected holding and backorder cost per time unit of RDC i in the approximate model, can be formulated as:

$$\begin{aligned}&C_i^A(S_i|\bar{L}_i(S_0)) \\ &\quad= \sum_{k=1}^{\mathrm{K}}\left(\frac{1}{Q_i}\sum_{u_i=0}^{Q_i-1}\sum_{m_{i,k}=0}^{u_i}\Pr(Z_i(u_i) = m_{i,k}) \times \lambda_{i,k}\hat{g}_{i,k}(S_{i,k} - m_{i,k}, \bar{L}_i(S_0))\right)\end{aligned} \tag{A.18}$$

According to Eq. (A.18), we can further obtain:

$$\begin{aligned}&C_i^A(S_i|\bar{L}_i(S_0)) \\ &= \sum_{k=1}^{\mathrm{K}}\left(\frac{1}{Q_i}\sum_{u_i=0}^{Q_i-1}\sum_{m_{i,k}=0}^{u_i}\left(\frac{u_i!}{m_{i,k}!(u_i-m_{i,k})!}\left(\frac{\lambda_{i,k}}{\Lambda_i}\right)^{m_{i,k}} \times \left(1-\frac{\lambda_{i,k}}{\Lambda_i}\right)^{u_i-m_{i,k}}\right) \times [(S_{i,k}-m_{i,k})(h_{i,k}+p_{i,k}) \\ &\quad\times F_{i,k}(S_{i,k}-m_{i,k}, \bar{L}_i) - \lambda_{i,k}\bar{L}_i(h_{i,k}+p_{i,k}) \\ &\quad\times F_{i,k}(S_{i,k}-m_{i,k}-1, \bar{L}_i) + p_{i,k}(\lambda_{i,k}\bar{L}_i - (S_{i,k}-m_{i,k}))]\right)\end{aligned} \tag{A.19}$$

where $F_{i,k}(S_{i,k} - m_{i,k}, \bar{L}_i) = 1 - \sum_{j=0}^{S_{i,k}-m_{i,k}-1}\frac{1}{j!}e^{-\lambda_{i,k}\bar{L}_i}(\lambda_{i,k}\bar{L}_i)^j$ and $F_{i,k}(S_{i,k} - m_{i,k} - 1, \bar{L}_i) = 1 - \sum_{j=0}^{S_{i,k}-m_{i,k}-2}\frac{1}{j!}e^{-\lambda_{i,k}\bar{L}_i}(\lambda_{i,k}\bar{L}_i)^j$.

The cost $C_i^A(S_i|\bar{L}_i(S_0))$ in Equation (A.19) is item-separable (item-additive), that is, it can be written as $C_i^A(S_i|\bar{L}_i(S_0)) = \sum_{k=1}^{K} C_{i,k}^A(S_{i,k}|\bar{L}_i(S_0))$, where

$$\begin{aligned}
&C_{i,k}^A(S_{i,k}|\bar{L}_i(S_{0,k}, k \in K)) \\
&= \frac{1}{Q_i} \sum_{u_i=0}^{Q_i-1} \sum_{m_{i,k}=0}^{u_i} \\
&\times \left(\frac{u_i!}{m_{i,k}!(u_i - m_{i,k})!} \left(\frac{\lambda_{i,k}}{\Lambda_i}\right)^{m_{i,k}} \left(1 - \frac{\lambda_{i,k}}{\Lambda_i}\right)^{u_i - m_{i,k}} \right) \\
&\times [(S_{i,k} - m_{i,k})(h_{i,k} + p_{i,k}) F_{i,k}(S_{i,k} - m_{i,k}, \bar{L}_i) \\
&- \lambda_{i,k}\bar{L}_i(h_{i,k} + p_{i,k}) F_{i,k}(S_{i,k} - m_{i,k} - 1, \bar{L}_i) \\
&+ p_{i,k}(\lambda_{i,k}\bar{L}_i - (S_{i,k} - m_{i,k}))] \qquad \text{(A.20)}
\end{aligned}$$

According to Axsäter (1993), $C_{i,k}^A(S_{i,k}|\bar{L}_i(S_0))$ is convex in $S_{i,k}$ for any given Q_i.

On the other hand, in the exact model where the replenishment lead-time of RDC i, L_i, is a random variable, we can formulate the expected cost of RDC i by conditioning on its lead-time. That is,

$$\begin{aligned}
&C_i^E(S_{i,k}|L_i(S_0)) \\
&= E_{L_i(S_0)}[\Pi_i(Q_i, S_{i,k})|L_i(S_0)] \\
&= E_{L_i(S_0)} \left[\sum_{k=1}^{K} \left(\frac{1}{Q_i} \sum_{u_i=0}^{Q_i-1} \sum_{m_{i,k}=0}^{u_i} \Pr(Z_i(u_i) = m_{i,k}) \right. \right. \\
&\left. \left. \times \ \lambda_{i,k}\hat{g}_{i,k}(S_{i,k} - m_{i,k}, l_i + L_i(S_0))|L_i(S_0) \right) \right] \qquad \text{(A.21)}
\end{aligned}$$

where $L_i(S_0)$ is a random lead time depending on S_0. However, it is difficult to evaluate the exact expected cost of RDC i by Eq. (A.21) since the probability distribution of $L_i(S_0)$ is hard to obtain.

The CDC cost function

As mentioned in Section 4.1, only shipping complete orders from the CDC to each RDC in the distribution system considered leads to the occurrence of reserved units. A reserved unit is ordered by a RDC and available at the CDC but it is not yet shipped. In case the on-hand inventory of the CDC is not enough to fill an entire RDC order in question, the units reserved by this order at the CDC must wait for the arrival of replenishments from the external supplier before they can be shipped.

The only cost component considered at the CDC is its inventory holding cost. Let $I_{0,k}^p$ denote on-hand inventory of item k at the CDC in case of partial order delivery, i.e. in case all reserved units at the CDC are delivered to the RDCs immediately without waiting for entire RDC order deliveries with other units from the external supplier. Then, the actual on-hand inventory of item k at the CDC in case of complete order delivery, denoted by $I_{0,k}$, is the sum of $I_{0,k}^p$ and $B_{0,k}^r$, i.e. $I_{0,k} = I_{0,k}^p + B_{0,k}^r$, where $B_{0,k}^r$ is the number of reserved units of item k at the CDC. Since the demand of the CDC is determined by the aggregate demand of all RDCs, we first have:

$$E(I_{0,k}^p) = S_{0,k} - L_0 \sum_{i=1}^{n} \lambda_{i,k} \qquad \text{(A.22)}$$

where $S_{0,k}$ is the order-up-to level of item k at the CDC.

Then, we have:

$$\begin{aligned}
C_0(S_0) &= \sum_{k=1}^{K} C_{0,k}(S_{0,k}) = \sum_{k=1}^{K} h_{0,k} E(I_{0,k}) \\
&= \sum_{k=1}^{K} h_{0,k}[E(I_{0,k}^p) + E(B_{0,k}^r)] \\
&= \sum_{k=1}^{K} h_{0,k} \left[S_{0,k} - L_0 \sum_{i=1}^{n} \lambda_{i,k} + E(B_{0,k}^r) \right] \qquad \text{(A.23)}
\end{aligned}$$

where $h_{0,k}$ is the holding cost per unit of item per unit of time for item k at the CDC.

Appendix C. Computation results

Please refer to this Appendix for detailed computation results of each instance set.