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Replacement and imperfect repair of deteriorating system: Study of a CBM policy and impact of repair efficiency

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Abstract:

This paper studies a condition-based predictive maintenance policy in the presence of imperfect repairs for deteriorating systems. The effect of imperfect repairs is assumed to be random in the sense that a maintenance action reduces the systems degradation level by a random proportion. The associated distribution allows to characterize the repair efficiency. As an example, the beta distribution is considered. It allows to define a wide range of repair efficiencies. A maintenance policy with an aperiodic and predictive inspection planning is considered and a new decision criterion is introduced to evaluate the extent to which a repair would be beneficial. The aim is to find a proper balance between repair and replacement for preventive action which leads to optimal maintenance cost. With this consideration, the cost model is developed from regenerative and semi-regenerative properties. The expected cost is evaluated and the optimal policy is reached as a function of repair efficiency. Finally, a numerical study is drawn up to demonstrate the implementation and highlight the potential benefits of the proposed methods. A sensitivity analysis on critical parameters is also performed.

Keywords: Condition-based maintenance, improvement factor model, imperfect maintenance, maintenance efficiency, inverse Gaussian process, optimization, Semi-Regenerative Process.

1 Introduction

Maintenance is referred to all actions done for preserving or restoring a system to its functioning mode. Among them, repairs play an essential role in bringing the system back from failure to a working state at a more reasonable cost than replacements. However, repairs are usually imperfect, which means they cannot restore the system to an “as good as new” state. Imperfect repairs can be affected by two factors: a tactical factor related to the type of maintenance action and an operational element that corresponds to the repairman’s expertise [15]. Different models have been proposed to evaluate the impact of imperfect repairs in maintenance. The most usual are virtual age models developed by Kijima [8], where repair is assumed to rejuvenate the system. The Brown-Proschan (BP) model introduced by [5] assumes that repair is either perfect or minimal with probabilities p and $1 - p$, respectively. The Geometric Process (GP) model proposed by Yeh [30] uses a non-increasing geometric process for consecutive repair times. The improvement factor model suggested by Malik[14] is based on the idea that imperfect repairs have different impacts, expressed by a coefficient multiplier. In this paper, we focus on the latter model; for reviewing more papers with these models, we may refer to [1, 13, 19, 21, 24, 25, 28].

Malik [14] introduced the concept of improvement factor to describe the imperfect repair effect in maintenance. In this model, an imperfect repair changes the hazard rate curve’s time to a more recent time but not to zero. The improvement factor model is powerful in the sense that just by adding one parameter it permits to have a variety of maintenance improvements, from none to full renewal. An extension of this model can be found in Lin et al. [11]. They proposed that imperfect maintenance has a more complicated effect, like changing both the time and the hazard rate slope. Related to such models, Chan and Shaw [2] suggest that each imperfect repair results in a reduction of failure rate, which depends on the system age and the number of preventive maintenance. Zequeira and Bérenguer [31] considered the case that the improvement factor depends on the time at which the preventive maintenance action are executed. Sheu and Chang [23] studied the case that preventive maintenance reduces the effective age and hazard rate distribution by employing two improvement factors. Zhou et al. [36] also introduced a hybrid model based on studies of Malik [14], and Nakagawa [18]. More recently, Zhang and Xie [33] developed an ameliorated improvement factor model for imperfect maintenance and its goodness of fit.

The improvement factor model's usage is not restricted to classical studies that deal with the system's lifetime and failure. It has been used to model the effect of imperfect repairs on degradation-based research. The influence of such maintenance actions is considered on various aspects, for example, on the critical threshold level, deterioration rate, and the amount of reduction of degradation level after each repair. Wang and Pham [27] applied the improvement factor for elevating the critical threshold level. They considered that imperfect preventive maintenance would improve the system immunity level. On the opposite, Shen et al. [22], with the help of the improvement factor model, considered the case that after each imperfect maintenance, the failure threshold lessens. It returns to the fact that a system's condition is not boosted as much as in previous maintenance actions, and hence the system will fail sooner. To name more physically meaningful studies in this concept, we may refer to Wang and Pham [26]. They considered a system subject to competing risks of degradation and random shocks and employed the improvement factor model to show the impact of imperfect repairs on the system's total damage. Zhang et al. [32] also proposed a model focused on changing the rate of deterioration of a system while each maintenance action could have a different degree of impact on the rate of deterioration. On the other hand, imperfect repairs usually recover the system to a better state. Zhao et al. [34] employed the improvement factor model to show this reduction of degradation level. They assessed the optimal imperfect repair policy by minimizing the expected warranty cost. Under the same consideration, Zhao et al. [35] treated the improvement factor as a random variable; hence each imperfect repair can affect diversely. Formulations other than the improvement factor are also exploited in connection with degradation processes. For example, Xu et al. [29] consider a random variable directly representing the degradation reduction.

Most of the efforts made in imperfect maintenance modeling aim at improving decision support systems. This paper focuses on degradation-based maintenance where perfect replacement and imperfect repair are available for preventive action. The problem of optimal balance between these two options is of particular importance for decision-making. The choice to be made depends strongly on the effectiveness of the repair. As an example, consider the performance of gas turbines. It always gradually decreases over time. The compressor has a significant impact on these performance losses. In particular, compressor fouling is a major concern. Foreign matter can cause pressure ratio drops that reduce power. Various maintenance actions can be considered to counter

the effects of fouling. Off-line water washing can be used to clean fouled blades. It results in partial cleaning with variable random effectiveness, which cannot be predicted precisely. It corresponds to an imperfect repair. Full maintenance efficiency can be achieved by mechanically cleaning the surfaces and possibly replacing damaged components. It requires opening the unit and corresponds to perfect maintenance action, hereafter referred to as “replacement”.

This work aims to address the issue of perfect and imperfect maintenance optimal configuration. A comprehensive modeling framework is proposed, including degradation modeling and imperfect maintenance modeling with joint description of its efficiency. Closed-form derivations of the cost are obtained, enabling efficient optimization of the decision variables. More specifically, this paper studies aperiodic condition-based predictive maintenance of a single unit system which is subject to degradation, when the system degradation evolution follows an inverse Gaussian (IG) process. Both replacement and repair are possible. To dynamically answer the crucial question of which one must be taken on-line, we introduce a criterion based on the probability of whether a repair could be proper. Adopting the same idea as [34, 35], the reduction of degradation level after an imperfect repair is modeled by an improvement factor, which is considered to have a beta distribution. The underlying stochastic processes are studied. Their properties are exploited to derive the long-term cost rate expression. The behavior of the proposed policy is studied numerically.

The rest of the paper is structured as follows. Section 2 provides general assumptions of the system, degradation modeling and improvement factor model for imperfect repair. Sections 3 and 4 respectively deal with maintenance policy description and analytical derivations for cost assessment. Sections 5 and 6 are devoted to the numerical study of the proposed policy. Finally, Section 7 concludes the paper with directions for further researches.

2 Terminology and assumptions

2.1 General assumptions

Consider a single unit system subjected to wear or deterioration. The degradation of the system is monotonically increasing over time. Continuous monitoring is not possible and the system condition can be observed only by inspection. It is assumed that inspections are instantaneous, non-destructive, and reveal the exact system state.

As the system's degradation level crosses a pre-specified threshold level L , failure happens and a corrective replacement must be carried out. Such replacements are perfect; i.e., the system's degradation will be restored to zero. A failure is not self-announced and can only be detected by inspection. Then, if the system fails between two inspections, it remains failed or unable to fulfill its requirements until the next inspection. This duration that results in an undesirable cost burden is referred to as downtime.

To avoid such unexpected occurrence of system failure, preventive maintenance is considered. It is associated with a threshold level on the degradation range, denoted as M . The preventive maintenance can be either repair or replacement. To be more realistic, repairs are assumed to be imperfect and should not reduce the degradation level to zero. As with corrective actions, preventive maintenance can only take place at inspection times. The cost of maintenance downtime due to such actions is included in the corresponding maintenance unit cost.

2.2 Degradation modeling

Without any repair or replacement actions, the system deterioration has an increasing continuous evolution. Let $X(t)$, with initial condition of $X(0) = 0$, stand for the system deterioration level at time $t \geq 0$.

Stochastic processes can be understandably used to model degradation data due to their time dependency structure. Here we assume that the underlying degradation follows an inverse Gaussian (IG) process. That means for $t \geq 0$ and $h > 0$, the increment $X_{t+h} - X_t$ is an inverse Gaussian distributed random variable with positive parameters μh and λh^2 . The pdf and cdf of such an increment are respectively:

$$f_h(x) = \sqrt{\frac{\lambda h^2}{2\pi x^3}} \exp\left\{-\frac{\lambda(x - \mu h)^2}{2\mu^2 x}\right\}, x > 0, \mu > 0, \lambda > 0,$$

and

$$F_h(x) = \Phi\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} - h\right)\right) + \exp\left(\frac{2\lambda h}{\mu}\right) \Phi\left(-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} + h\right)\right),$$

where Φ is the cdf of the standard normal distribution, and the subscript h indicates the length of interval $(t, t + h)$ associated with the increment.

2.3 Imperfect repair modeling

To describe the effect of imperfect repair, a random improvement factor model is adopted. That means the degradation level of the system after an imperfect preventive action at time T_n is:

$$X_{T_n} = \delta \cdot X_{T_n}^-,$$

where $0 < \delta < 1$ and $X_{T_n}^-$ is the degradation level of the system just before the maintenance action. We assume that δ is a random variable to characterize the randomness in the effect of improvement. For this purpose, the beta distribution with shape parameters α and β is considered. Its pdf (resp. cdf) is denoted q (resp. Q). As a consequence:

$$q(\delta) = \frac{\delta^{\alpha-1}(1-\delta)^{\beta-1}}{B(\alpha, \beta)} \text{ and } Q(\delta) = \frac{B(\delta; \alpha, \beta)}{B(\alpha, \beta)} = I_\delta(\alpha, \beta)$$

where $B(\delta; \alpha, \beta)$ is the incomplete beta function and $I_\delta(\alpha, \beta)$ is the regularized incomplete beta function. The possible shapes of beta distributions are decreasing, increasing, U curved, inverted U curved, or even constant. The main reason for the choice of the beta distribution as the distribution of improvement factor is its ability to take this great diversity of shapes using only two parameters. Depending on the values of the two parameters, the density function can be symmetric ($\alpha = \beta$), left-skewed ($\alpha > \beta$), and right-skewed ($\alpha < \beta$). Figure 1 depicts pdf of beta distribution for some values of α and β along with their cdf and quantile function.

With this consideration, the mean and variance of the random improvement factor δ are:

$$\mathbb{E}(\delta) = \frac{\alpha}{\alpha + \beta}, \quad \text{and} \quad \mathbb{V}(\delta) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.$$

These values can fully represent the efficiency characteristics of imperfect repairs. As the parameters α and β are such that $\mathbb{E}(\delta)$ is near to zero, imperfect maintenance can be regarded as an efficient repair in sense that the degradation level after repair will be very low. Similarly, higher amount of $\mathbb{E}(\delta)$ is associated with a poor or an inefficient repair. The value of $\mathbb{V}(\delta)$ is related to consistency of repair actions.

3 Maintenance policy

When a system fails, replacing it is the inevitable corrective action. But when the system is still running, and its degradation level is higher than M , choosing rationally between replacement and

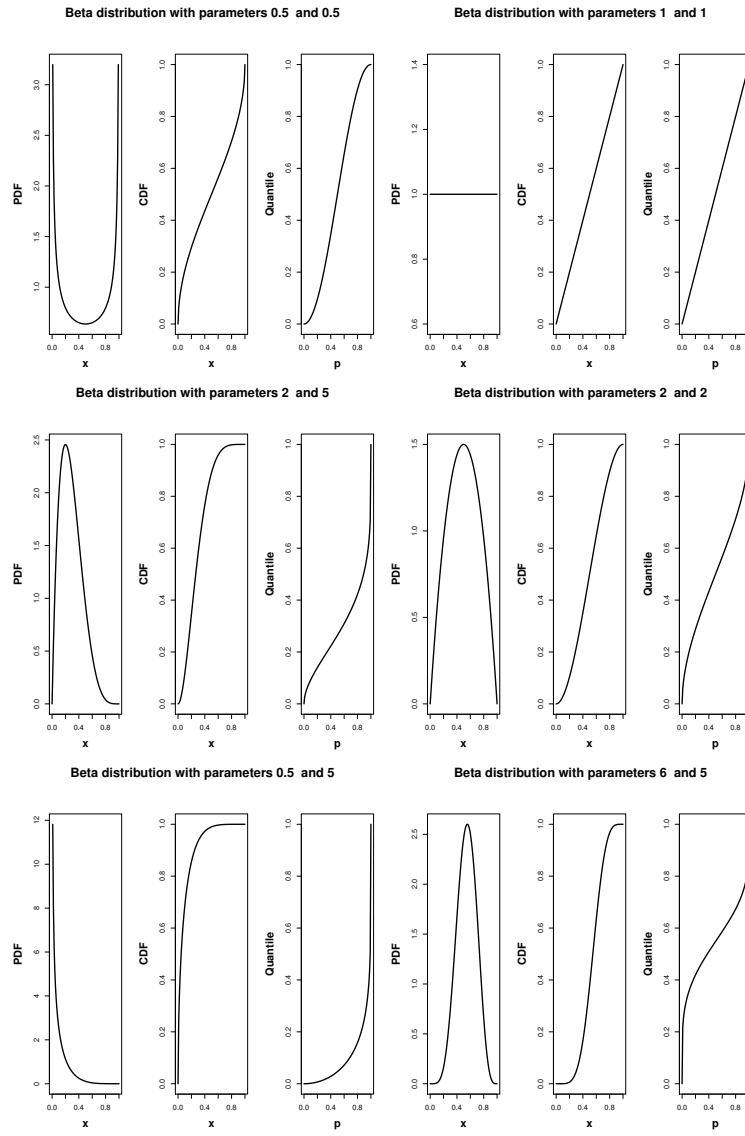


Figure 1: Various shapes of pdf, cdf and quantile function of beta distribution

repair is one of the major concerns. Repair is cheaper, but the effect produced by a repair cannot be predicted with certainty. Unlike a replacement, the degradation level and, therefore, the expected time to failure of a newly repaired unit cannot be guaranteed. As the maintenance cost is high, we are looking for a trade-off between repairs and replacements.

To decide what should be done at each inspection, first we introduce φ which is the probability to leave the preventive zone after an repair. It is defined as the function $x \mapsto \varphi(x) = \mathbb{P}(\delta.x \geq M)$ for $x \in [M, L]$. Knowing the distribution of δ , It can be rewritten as:

$$\varphi(x) = 1 - Q\left(\frac{M}{x}\right).$$

It is worthy to note that $\varphi(\cdot)$ is an increasing function on its domain and then it is invertible.

3.1 Inspection times

It is essential to choose a specific structure to determine inspection times. This time scheduling may be periodic or aperiodic. Although, the former is practically easier, the latter helps us to reduce the cost by preventing unnecessary frequent inspections. Moreover, by choosing aperiodic inspection planning which considers the current situation of the system, the risk of system failure reduces. Here, we adopt the RUL-based inspection planning.

The RUL is defined as the remaining duration for which a system will work before it fails given its current state. If the current degradation level of the system at time t is x , then the remaining useful life of the system, Υ_x , is defined as:

$$\Upsilon_x = \inf\{r > 0 : X_{t+r} \geq L | X_t = x\}.$$

Considering the IG process as the underlying process, the cdf of RUL is:

$$G_{\Upsilon}(r|x) = \Phi\left(\sqrt{\frac{\lambda}{L-x}}\left(r - \frac{L-x}{\mu}\right)\right) - \exp\left(\frac{2\lambda r}{\mu}\right) \Phi\left(-\sqrt{\frac{\lambda}{L-x}}\left(r + \frac{L-x}{\mu}\right)\right),$$

where Φ is the cdf of the standard normal distribution.

Then, denoting the sequence of inspection times by $\{T_n\}_{n \in \mathbb{N}}$ with the initial value of $T_0 = 0$, the RUL-based inspection scheduling is derived by:

$$T_{n+1} = T_n + \tau_p(X_{T_n}),$$

where $\tau_p(X_{T_n})$ is the p -quantile of the RUL distribution; i.e:

$$\tau_p(X_{T_n}) = \{\Delta t : G_{\Upsilon}(\Delta t | X_{T_n}) = p\} \quad (3.1)$$

3.2 Decision rule

In case of maintenance at time T_n , let denote T_n^- the time just before the maintenance. As a consequence $X_{T_n^-}$ is the degradation level at time T_n just before the maintenance action and X_{T_n} the degradation level after maintenance. The policy is as follows:

- If $X_{T_n^-} \geq L$, a corrective replacement is taken which restore system to "as good as new" state. That means after maintenance $X_{T_n} = 0$;
- If $M \leq X_{T_n^-} < L$, a preventive maintenance must be performed. In order to choose between perfect or imperfect maintenance, we employ the function $\varphi(\cdot)$ as follows:
 - if $\varphi(X_{T_n^-}) \geq s$, then the effectiveness of repair would not be sufficient, and perfect preventive maintenance is taken. The system is replaced and the its degradation level will be restored to zero; i.e. $X_{T_n} = 0$;
 - if $\varphi(X_{T_n^-}) < s$, then a repair is chosen. The maintenance is imperfect and $X_{T_n} = \delta X_{T_n^-}$. If after imperfect maintenance $X_{T_n} \geq M$ then a perfect action is performed immediately such that $X_{T_n} = 0$. It happens when, contrary to what was expected, the reparation effect has been insufficient. As the degradation level after the repair is required, an inspection has to be performed. In the following, this inspection is part of the maintenance action, i.e., a degradation level measurement systematically follows the repair. Its cost is included in the unitary repair cost.
- If $0 \leq X_{T_n^-} < M$, the system is properly working and no maintenance is needed. Maintenance decision-making is postponed to the next inspection time.

4 Stationary law and cost

This section deals with the evaluation of the long-run average cost per time unit without simulation and with the help of semi-regenerative properties. By elementary renewal theory, it is known that the long-run average cost per time unit is equal to the ratio of the expected cost on the first renewal cycle, over the expected length of that cycle for almost any realization of the process. However, expectations over a renewal cycle are tricky in case of aperiodic inspections. To assess the cost function, we will follow the results developed in Coccozza-Thivent [3]. It is shown that the

long-run average cost rate can be changed into a ratio of expectations on one semi-renewal cycle. Considering a shorter semi-renewal cycle instead of a whole renewal cycle, as applied in [4, 7, 17], is more favorable. This method first requires characterizing the stationary behavior of the maintained system at the inspection times and then calculating the cost by integration with respect to the stationary law.

For this aim, let $\{Y_n = X_{T_n}\}_{n \in \mathbb{N}}$ be the discrete-time random process describing the system state at each inspection time, after the possible maintenance action has been carried out. The properties of the process $\{X_t\}_{t \geq 0}$ and the embedded chain $\{Y_n\}_{n \in \mathbb{N}}$, when $\{X_t\}_{t \geq 0}$ is monotone increasing stochastic process is listed in [17] for both case of known and unknown model parameters. In the following subsections, we first derive the expression of stationary law, and then present the cost function.

4.1 Expression of stationary law

Primarily, we derive the transition probability distribution function of $\{Y_n\}$ given the current state. It is worthy to note that it is a mixture distribution since, given its value at time T_n , the system's state Y_{n+1} at time T_{n+1} is a random variable which should be exactly 0 in case of perfect maintenance or any value $y \in (0, M)$ otherwise. The former case is the result of a corrective replacement after failure, or the result of a preventive replacement when the system degradation level increases to a value between M and L during the time interval $[T_n, T_{n+1})$. The latter case is the outcome of an imperfect repair, or it means that the system is still in good condition at time T_{n+1} . Due to replacements, the system degradation level can be restored to the value 0 with a non-zero probability, which is not the case for other degradation values. Consequently, the transition probability of the Markov chain $\{Y_n, n \in \mathbb{N}\}$ given the current state involves mixed variables. To assess this transition probability distribution, we evaluate the two different probabilities:

- In case of a replacement at time T_{n+1} (discrete part):

$$\begin{aligned} \mathbb{P}(Y_{n+1} = 0 | Y_n = x) \\ = \mathbb{P}\left(\left\{Y_{n+1}^- \geq L\right\} \text{ or } \left\{(M \leq Y_{n+1}^- < L) \text{ and } (\varphi(Y_{n+1}^-) \geq s)\right\} \text{ or } \right. \\ \left. \left\{(M \leq Y_{n+1}^- < L) \text{ and } (\varphi(Y_{n+1}^-) < s) \text{ and } (\delta \cdot Y_{n+1}^- \geq M)\right\} | Y_n = x\right), \end{aligned}$$

with $Y_{n+1}^- = X_{T_{n+1}^-}$ the degradation level at time T_{n+1} just before maintenance action.

By defining $Z = Y_{n+1}^- - Y_n$ which has the distribution $IG(\mu, \tau_p(x), \lambda, \tau_p^2(x))$ and $\omega = \varphi^{-1}(s) \in [M, L]$, it can be rewritten as:

$$\begin{aligned}
& \mathbb{P}(Y_{n+1} = 0 | Y_n = x) \\
&= \mathbb{P}\left(\left\{Z \geq L - x\right\} \text{ or } \left\{(M - x \leq Z < L - x) \text{ and } (\varphi(x + Z) \geq s)\right\} \text{ or } \right. \\
&\quad \left. \left\{(M - x \leq Z < L - x) \text{ and } (\varphi(x + Z) < s) \text{ and } (\delta.(x + Z) \geq M)\right\}\right) \\
&= \mathbb{P}\left(Z \geq L - x\right) + \mathbb{P}\left((M - x \leq Z < L - x) \text{ and } (Z \geq \omega - x)\right) \\
&\quad + \mathbb{P}\left((M - x \leq Z < L - x) \text{ and } (Z < \omega - x) \text{ and } (\delta.(x + Z) \geq M)\right) \\
&= \mathbb{P}(Z \geq L - x) + \mathbb{P}(\max(M, \omega) - x \leq Z < L - x) \\
&\quad + \mathbb{P}\left((M - x \leq Z < \min(L, \omega) - x) \text{ and } (\delta \geq \frac{M}{x + Z})\right) \\
&= \bar{F}_{\tau_p(x)}(\omega - x) + \int_{M-x}^{\omega-x} \bar{Q}\left(\frac{M}{x+z}\right) f_{\tau_p(x)}(z) dz, \tag{4.1}
\end{aligned}$$

where, $\bar{F}_{\tau_p(x)}$ and \bar{Q} are survival functions of IG distribution with parameters $\mu, \tau_p(x)$ and $\lambda, \tau_p^2(x)$ and beta distribution with shape parameters α and β .

- In case of a repair or without any maintenance action at time T_{n+1} (continuous part):

$$\begin{aligned}
& \mathbb{P}(Y_{n+1} \leq y | Y_n = x) \\
&= \mathbb{P}\left((M \leq Y_{n+1}^- < L) \text{ and } (\varphi(Y_{n+1}^-) < s) \text{ and } (\delta.Y_{n+1}^- \leq y) | Y_n = x\right) \cdot \mathbb{I}_{\{y < M\}} \\
&\quad + \mathbb{P}\left(Y_{n+1}^- \leq y | Y_n = x\right) \cdot \mathbb{I}_{\{x \leq y < M\}} \\
&= \mathbb{P}\left((M - x \leq Z < L - x) \text{ and } (\varphi(x + Z) < s) \text{ and } (\delta.(x + Z) \leq y)\right) \cdot \mathbb{I}_{\{y < M\}} \\
&\quad + \mathbb{P}\left(Z \leq y - x\right) \cdot \mathbb{I}_{\{x \leq y < M\}} \\
&= \mathbb{P}\left((M - x \leq Z < L - x) \text{ and } (Z < \omega - x) \text{ and } (\delta \leq \frac{y}{x + Z})\right) \cdot \mathbb{I}_{\{y < M\}} \\
&\quad + \mathbb{P}\left(Z \leq y - x\right) \cdot \mathbb{I}_{\{x \leq y < M\}} \\
&= \int_{M-x}^{\omega-x} Q\left(\frac{y}{x+z}\right) f_{\tau_p(x)}(z) dz \cdot \mathbb{I}_{\{y < M\}} + F_{\tau_p(x)}(y - x) \cdot \mathbb{I}_{\{x \leq y < M\}}. \tag{4.2}
\end{aligned}$$

From the transition probabilities, i.e. from equation (4.1) and by derivation of equation (4.2), the probability distribution is written as the sum of a discrete part with the Dirac delta function

and a density part:

$$p(y|x) = A_1(x)\delta_0(y) + \left(A_2(x, y)\mathbb{I}_{\{y < M\}} + f_{\tau_p(x)}(y - x)\mathbb{I}_{\{x \leq y < M\}} \right), \quad (4.3)$$

where $A_1(x)$ and $A_2(x)$ are respectively given by:

$$\begin{aligned} A_1(x) &= \overline{F}_{\tau_p(x)}(\omega - x) + \int_{M-x}^{\omega-x} \overline{Q}\left(\frac{M}{x+z}\right) f_{\tau_p(x)}(z) dz, \\ A_2(x, y) &= \frac{\partial}{\partial y} \left(\int_{M-x}^{\omega-x} Q\left(\frac{y}{x+z}\right) f_{\tau_p(x)}(z) dz \right) \\ &= \int_{M-x}^{\omega-x} \frac{1}{x+z} f_{\tau_p(x)}(z) q\left(\frac{y}{x+z}\right) dz, \end{aligned}$$

and the Dirac delta function δ_0 is such that $\delta_0(x) = 0$ for $x \neq 0$ and $\int f(x)\delta_0(x)dx = f(0)$ (see [10] for more properties of this function).

Let π denote the stationary law of the Markov chain $\{Y_n\}$. As for the transition probability distribution of $\{Y_n\}$, the stationary law π is a convex combination of a Dirac delta function and a density function. It comes, with $0 < a < 1$:

$$\pi(x) = a\delta_0(x) + (1-a)b(x). \quad (4.4)$$

By definition, the stationary law is the solution of stationary equation:

$$\pi(y) = \int_0^M p(y|x)\pi(x)dx. \quad (4.5)$$

By substituting (4.3) and (4.4) in (4.5), it comes respectively from the Dirac part and from the density part:

$$a = aA_1(0) + (1-a) \int_0^M A_1(x)b(x)dx, \quad (4.6)$$

and

$$b(y) = \frac{a}{1-a} \left[A_2(0, y) + f_{\tau_p(0)}(y) \right] + \int_0^M A_2(x, y)b(x)dx + \int_0^y f_{\tau_p(x)}(y-x)b(x)dx. \quad (4.7)$$

The stationary law can be assessed by a numerical integration as follows. First, equation (4.7) can be multiplied by $\frac{1-a}{a}$ and can be written:

$$B(y) = A_2(0, y) + f_{\tau_p(0)}(y) + \int_0^M A_2(x, y)B(x)dx + \int_0^y f_{\tau_p(x)}(y-x)B(x)dx \quad (4.8)$$

with $B(y) = \frac{1-a}{a}b(y)$.

As a solution of the previous integral equation, the function $B(y)$ can be evaluated numerically through the method introduced in Appendix A.

Then, from $\int \pi(x)dx = 1$ and substituting π from (4.4), the value of a is obtained with

$$a = \frac{1}{1 + \int_0^M B(x)dx}. \quad (4.9)$$

Finally, from a and $B(y)$ it comes:

$$b(y) = \frac{a}{1-a} B(y). \quad (4.10)$$

Fig. 2 shows the stationary law for two examples. A simulation study is also conducted to insure the accuracy of the method.

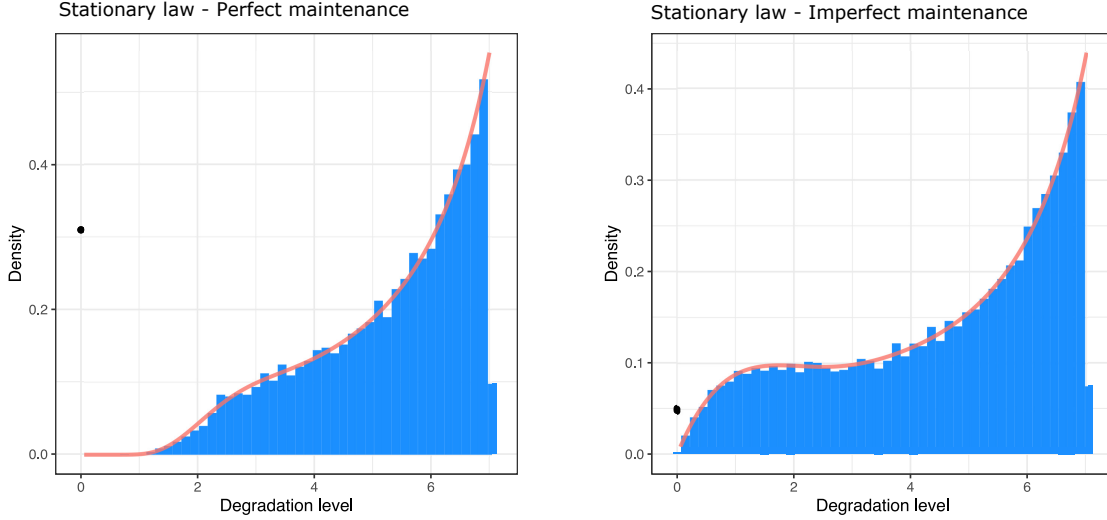


Figure 2: The Markov chain's stationary law: comparison of numerical solution of integral equation and histogram from Monte Carlo simulation. The two examples with perfect maintenance ($s = 0$) on the left and imperfect maintenance ($s = \varphi(L)$) on the right. The degradation parameters are $\mu = 1, \lambda = 1$, shape parameters of the beta distribution are $\alpha = 2, \beta = 5$, and the failure level is $L = 9$. Decision variables are fixed as $M = 7$ and $p = 0.05$. The black dot is the value of a in (4.9).

4.2 Maintenance cost

Each action incurs a unitary cost. We consider that C_i, C_r, C_{cr} , and C_{pr} are respectively the inspection cost, the repair cost, the cost of corrective replacement, and the cost of preventive replacement. Let C_{r+} denote the unit cost in case that a perfect maintenance action is inevitable

after an imperfect maintenance action. It is obtained from an additional preventive cost C_{add} such that $C_{r+} = C_r + C_{\text{add}}$. It is reasonable to suppose that:

$$C_i < C_r < C_{\text{pr}} \leq C_{\text{cr}} \leq C_{r+}.$$

The cost rate in case of unavailability is also assumed to be C_d . With this consideration, the cumulative cost function on $[0, t]$ is:

$$C(t) = C_i N_i(t) + C_r N_r(t) + C_{\text{pr}} N_{\text{pr}}(t) + C_{\text{cr}} N_{\text{cr}}(t) + C_{r+} N_{r+}(t) + C_d d(t),$$

where $N_i(t)$, $N_{\text{cr}}(t)$, $N_{\text{pr}}(t)$, and $N_r(t)$ are the number of inspections, the number of corrective replacements, the number of preventive replacements, and the number of repairs, respectively. The number of repairs followed by a replacement is showed by $N_{r+}(t)$. The duration that the system passed in downtime until t is denoted by $d(t)$.

The aim is optimizing the long-run average cost per unit of time, i.e. finding the values of decision variables p , s and M which minimizes

$$EC = \lim_{t \rightarrow \infty} \frac{C(t)}{t}.$$

As mentioned in [3], the semi regenerative properties of the degradation process help us to assess this limit as follows

$$EC = \frac{\mathbb{E}_\pi(C(T_1))}{\mathbb{E}_\pi(T_1)},$$

where the $\mathbb{E}_\pi(\cdot)$ refers to expectation with respect to the stationary law π . Therefore,

$$EC = \frac{1}{\mathbb{E}_\pi(T_1)} \left\{ C_i \mathbb{E}_\pi(N_i(T_1)) + C_r \mathbb{E}_\pi(N_r(T_1)) + C_{\text{pr}} \mathbb{E}_\pi(N_{\text{pr}}(T_1)) + \right. \\ \left. C_{\text{cr}} \mathbb{E}_\pi(N_{\text{cr}}(T_1)) + C_{r+} \mathbb{E}_\pi(N_{r+}(T_1)) + C_d \mathbb{E}_\pi(d(T_1)) \right\}. \quad (4.11)$$

The expectations can be assessed as follows:

$$\begin{aligned} \mathbb{E}_\pi(N_i(T_1)) &= 1, \\ \mathbb{E}_\pi(N_{\text{cr}}(T_1)) &= P_\pi(X_{T_1^-} > L) = \int_0^M \bar{F}_{\tau_p(x)}(L - x) \pi(x) dx, \\ \mathbb{E}_\pi(T_1) &= \int_0^M \tau_p(x) \pi(x) dx. \end{aligned}$$

where, as introduced in section 4.1, $\bar{F}_s(z) = \mathbb{P}(X_s \geq z)$ is the survival functions of IG distribution with parameters $\mu.s$ and $\lambda.s^2$, which corresponds to the degradation increment over a time interval of width s .

Moreover, the expectation for number of perfect preventive actions is:

$$\begin{aligned}\mathbb{E}_\pi(N_{\text{pr}}(T_1)) &= P_\pi(M \leq X_{T_1^-} \leq L \text{ and } \varphi(X_{T_1^-}) \geq s) \\ &= \int_0^M \mathbb{P}_x(M < x + Z \leq L; x + Z \geq \omega) \pi(x) dx \\ &= \int_0^M [F_{\tau_p(x)}(L - x) - F_{\tau_p(x)}(\omega - x)] \pi(x) dx,\end{aligned}$$

the expectation for number of imperfect preventive actions is:

$$\begin{aligned}\mathbb{E}_\pi(N_{\text{r}}(T_1)) &= P_\pi(M \leq X_{T_1^-} \leq L, \text{ and } \delta X_{T_1^-} < M \text{ and } \varphi(X_{T_1^-}) < s) \\ &= \int_0^M \mathbb{P}\left(M - x \leq Z < L - x; Z < \frac{M}{\delta} - x; Z < \omega - x\right) \pi(x) dx \\ &= \int_0^M \mathbb{P}\left(M - x \leq Z < \min(\omega, \frac{M}{\delta}) - x\right) \pi(x) dx \\ &= \int_0^M \left\{ \int_0^1 \int_{M-x}^{\min(\omega, \frac{M}{\delta})-x} f_{\tau_p(x)}(z) q(\delta) dz da \right\} \pi(x) dx \\ &= \int_0^M \left(\int_{M-x}^{\omega-x} Q\left(\frac{M}{x+z}\right) f_{\tau_p(x)}(z) dz \right) \pi(x) dx,\end{aligned}$$

and the expectation for imperfect preventive action followed by a perfect one can be calculated as:

$$\begin{aligned}\mathbb{E}_\pi(N_{\text{r}+}(T_1)) &= P_\pi(M \leq X_{T_1^-} \leq L, \text{ and } \delta X_{T_1^-} \geq M \text{ and } \varphi(X_{T_1^-}) < s) \\ &= \int_0^M \mathbb{P}\left(M - x \leq Z < L - x; Z \geq \frac{M}{\delta} - x; Z < \omega - x\right) \pi(x) dx \\ &= \int_0^M \mathbb{P}\left(M - x \leq Z < L - x; \delta \geq \frac{M}{x+Z}; Z < \omega - x\right) \pi(x) dx \\ &= \int_0^M \left(\int_{M-x}^{\omega-x} \bar{Q}\left(\frac{M}{x+z}\right) f_{\tau_p(x)}(z) dz \right) \pi(x) dx.\end{aligned}$$

Let finally express the expected unavailability duration over a single semi-renewal cycle. It is defined as

$$\mathbb{E}_\pi(d(T_1)) = \int_0^M \mathbb{E}_x(d_1) \pi(x) dx$$

where $\mathbb{E}_x(d_1)$ is the mean unavailability time knowing the degradation level is x at the beginning of the cycle. The next inspection time is $\tau_p(x)$ time units later. As a consequence,

$$\begin{aligned}\mathbb{E}_x(d_1) &= \mathbb{E}\left(\int_0^{\tau_p(x)} \mathbb{I}_{\{X_s \geq L-x\}} ds\right) \\ &= \int_0^{\tau_p(x)} \mathbb{E}_x(\mathbb{I}_{\{X_s \geq L-x\}}) ds = \int_0^{\tau_p(x)} \mathbb{P}(X_s \geq L-x) ds\end{aligned}$$

and

$$\mathbb{E}_\pi(d(T_1)) = \int_0^M \left[\int_0^{\tau_p(x)} \bar{F}_s(L-x) ds \right] \pi(x) dx.$$

After developing the expression of the maintenance long-run cost rate, the next two sections are devoted to a numerical study of the proposed maintenance policy hereafter referred to as “(p,M,s)-policy”. The aim is to investigate the maintenance cost function, discuss its optimization, and analyze the influence of unit maintenance costs.

5 Numerical optimization and performance assessment

Different numerical experiments are presented to illustrate some useful properties and characterize optimal decision variables. From now and for the sake of the paper clarity, the efficiency of imperfect maintenance, i.e, the behavior of the random improvement factor δ , is characterized only by the first shape parameter α of the beta distribution and the second parameter β is fixed and equal to 5. Thus a low (resp. high) value of α means “efficient” (resp. “inefficient”) repair.

The effectiveness of a condition-based or predictive maintenance policy relies heavily on identifying relevant decision variables. To this end, the optimization of the evaluation criterion is a crucial phase. According to the proposed maintenance policy, it consists of finding decision variables (s^*, p^*, M^*) such that:

$$EC^* = EC(s^*, p^*, M^*) \leq EC(s, p, M), \quad \forall (s, p, M) \in [0, \varphi(L)] \times [0, 1] \times [0, L].$$

The maintenance cost rate is a continuous function. The constraint set is closed and convex. Then the optimization problem has a solution. The choice of the optimization algorithm may depend on the shape of the cost function. It can be different in case of the existence of local minima or not. The analysis of the cost shape is addressed first. The second part of this section is related to the decision variable for preventive repair or replacement.

Numerous tests have been conducted to exhibit cost function behavior. The examples given hereafter are representative of the results obtained. They present the general observed characteristics. The degradation parameters are $\mu = 1$ and $\lambda = 1$, and the failure level is $L = 9$. Throughout this section, unless otherwise specified, the following unit costs are chosen:

$$C_i = 0.2, C_r = 4, C_{pr} = 7, C_{cr} = 10, C_{r+} = 11 \text{ (or } C_{add} = 7), C_d = 4 \quad (5.1)$$

5.1 Cost shape and optimization

Due to the lack of tractable closed-form expression, it is difficult to obtain a general proof of unicity of a cost minimum. However, such a property is of particular interest for numerical minimization and practical search of optimal maintenance variables.

First, the maintenance cost rate's convexity as a function of p and M is highlighted. To this end, four illustrative examples of level curves are given with fixed values of decision variable s . The examples are established as follows: two different beta distributions describe repair efficiency. The two beta distributions represent successively the cases of a rather efficient repair with parameter $\alpha = 2$, i.e. $\mathbb{E}[\delta] \simeq 0.286$, and a less efficient repair with $\alpha = 5$, i.e. $\mathbb{E}[\delta] \simeq 0.5$. For each case of efficiency, two values of decision variable s are considered. The extreme values $s = \varphi(M)(= 0)$ and $s = \varphi(L)$ are considered to illustrate. The former choice of s leads to the case that only replacement is taken as preventive maintenance, while the latter choice is related to the case that only repair is done for preventive actions.

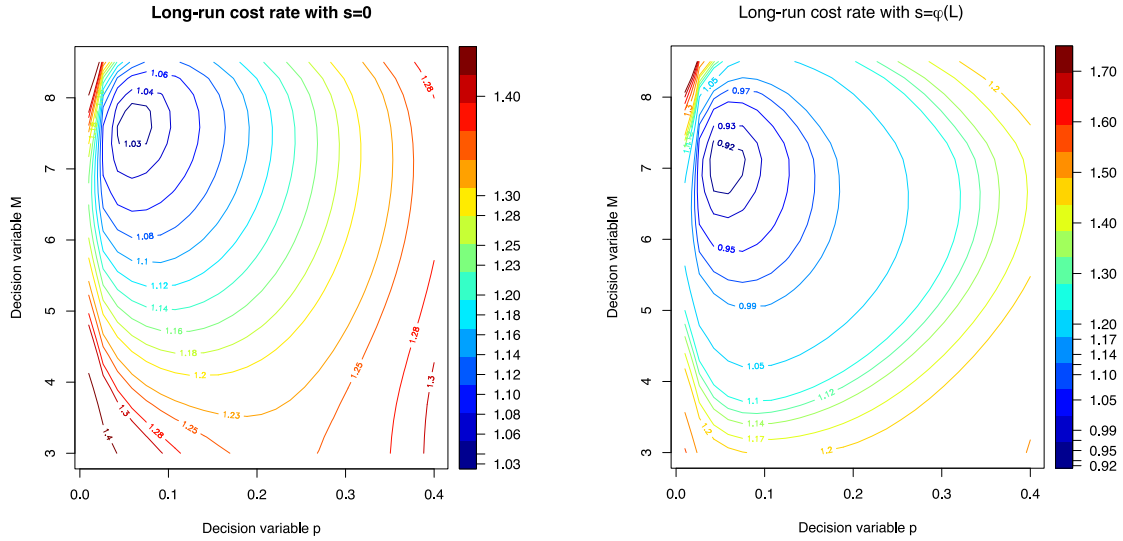


Figure 3: Maintenance cost as a function of p and M for fixed values of s such that $s = \varphi(M) = 0$, i.e., perfect maintenance actions (left) and $s = \varphi(L)$, i.e., imperfect actions (right). Rather efficient repairs, $\delta \sim \text{Beta}(2, 5)$, is considered.

The cost functions are depicted in Figures 3 and 4 for the most efficient and the less efficient repair cases, respectively. As highlighted in these figures, the level curves of maintenance cost are convex functions for a fixed value of s . It can be seen that the optimal values of p and M are

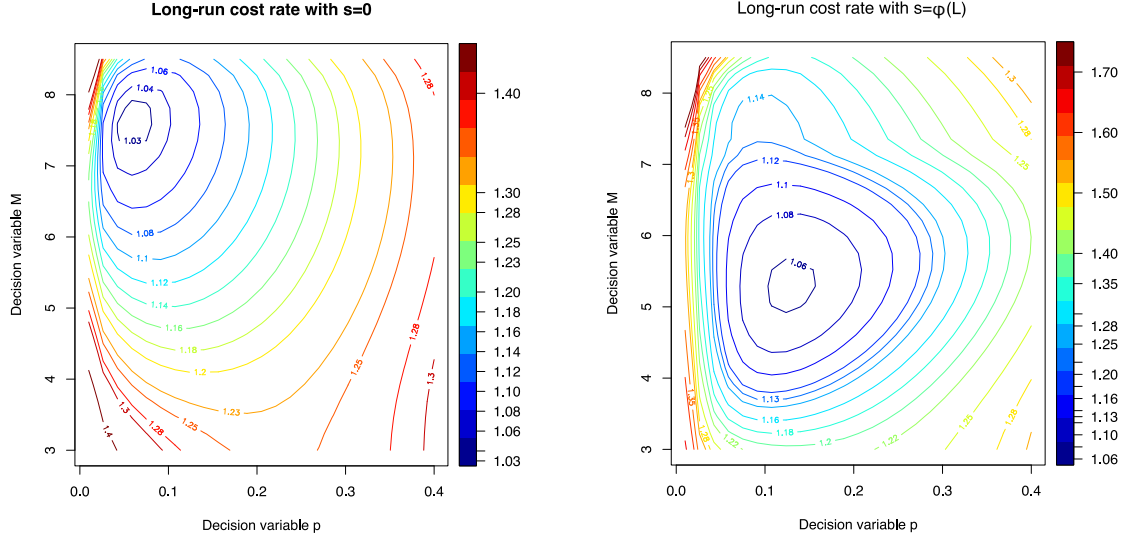


Figure 4: Maintenance cost as a function of p and M for fixed values of s such that $s = \varphi(M)$, i.e., perfect maintenance actions (left) and $s = \varphi(L)$, i.e., imperfect actions (right). Less efficient repairs, $\delta \sim \text{Beta}(5, 5)$, is considered.

different when s equals 0 or $\varphi(L)$. For the high value of s , which means more repairs in preventive actions, M^* is lower, and p^* is higher. Moreover, for a given repair efficiency, the magnitude of the difference between the configurations corresponding to the two values of s strongly dependent on the efficiency level α . That means the difference in optimal values is not significant as long as α remains small i.e. when the repairs are efficient. It is related to the fact that efficient repairs reduce the system's degradation near to zero, almost the same as for replacements. Hence the difference between configurations with repair and with preventive replacement is not significant.

Let secondly consider the maintenance cost as a function of s with fixed values of decision variables p and M . Possible evolution of EC is depicted on Figure 5 for two configurations of beta distribution. In this figure, s takes values between its bounds 0 and $\varphi(L)$.

As previously, preventive repairs are respectively such that α is equal to 2 and 5. The fixed decision variables p and M have been set to their corresponding optimal values p^* and M^* . In the first case, with $\alpha = 2$, the optimal value s^* is equal to $\varphi(L)$, which corresponds to preventive actions with only repair and no replacement. Conversely, in the second case $s^* = 0$, which means only replacements and no repair for preventive actions. Intermediate values of α lead to optimal configurations with a balance between repairs and replacements.

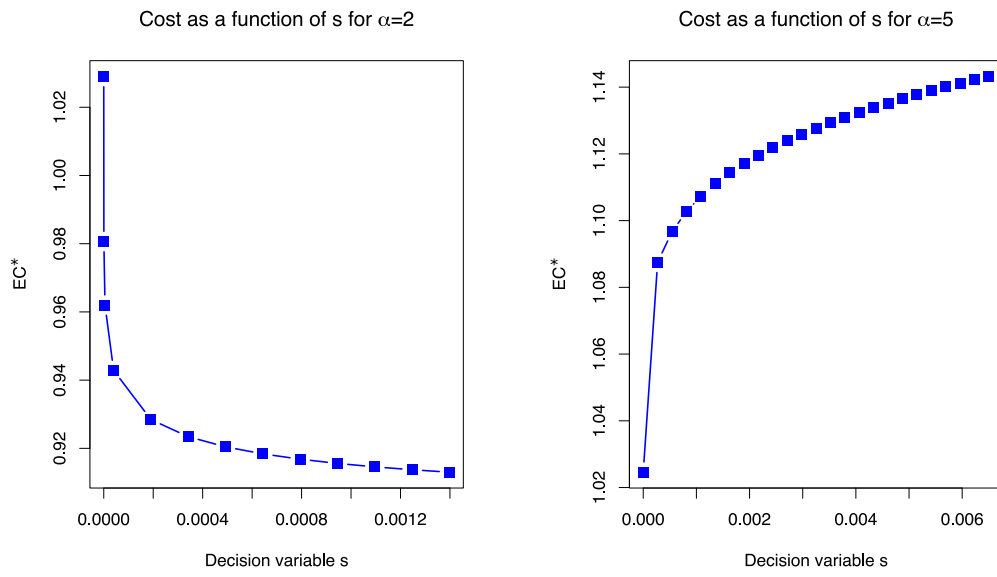


Figure 5: Maintenance cost as a function of s for fixed values of p and M . The two cases are associated with rather efficient repairs i.e., $\delta \sim \text{Beta}(2, 5)$ (left) and less efficient repairs with $\delta \sim \text{Beta}(5, 5)$ (right). The values of p and M are the optimal values for each case i.e. respectively $(p, M) = (0.53, 7.25)$ and $(0.57, 7.6)$.

For all tested configurations, there is a unique global minimum and no local minima. Consequently, proper algorithms developed for continuous non-linear optimization allow finding the set of decision variables that minimize the maintenance cost rate. Quasi-Newton (BFGS) or Nelder-Mead methods can be used to obtain numerical results. As it does not require derivative calculation, Nelder-Mead with the log-barrier method for constraints (see for example [6]) is considered to obtain the results given in this paper. The algorithm for cost calculation is given in Appendix B.

5.2 Performance assessment in comparison to classical policies

The (p,M,s)-policy considers the decision variable s , which is a limit value on the probability that the degradation level remains greater than M after repair. It is a specificity compared to other policies. Without it, preventive maintenance actions are all of the same kind, should it be imperfect repairs or perfect replacements. The classical decision rules can be considered as particular cases of the proposed rule with s . For example, a perfect preventive policy corresponds to $s = 0$ and an imperfect preventive policy to $s = \varphi(L)$. Thus, the proposed policy is always better than a classical imperfect or perfect preventive policy. It allows for finer decision making. This section analyzes this strategy's performance to find a balance between perfect and imperfect preventive maintenance actions.

Figure 6 depicts comparisons of (p,M,s)-policy with fully perfect and imperfect ones. The relative increase of classical policies costs compared to the proposed one are given. It shows that the introduction of the new decision variable s can allow a significant gain.

More precisely, for different repair efficiency cases, the optimal cost of the (p,M,s)-policy is compared on one side with the optimal cost of a perfect preventive policy and on the other side with the optimal cost of an imperfect preventive policy. The evolution of imperfect maintenance efficiency is described with parameter α of the beta distribution varying from 0.5 to 6, i.e., for $\mathbb{E}[\delta]$ increasing from 0.09 to 0.55. The results are given as a percentage of (p,M,s)-policy's optimal cost, which is always the lowest. In most efficient repairs, i.e., for $\alpha = 0.5$, the proposed policy is an imperfect preventive policy and is 30% more efficient than a perfect preventive policy. On the opposite, in case of less efficient imperfect repairs, i.e., for $\alpha = 6$, the proposed policy acts as a

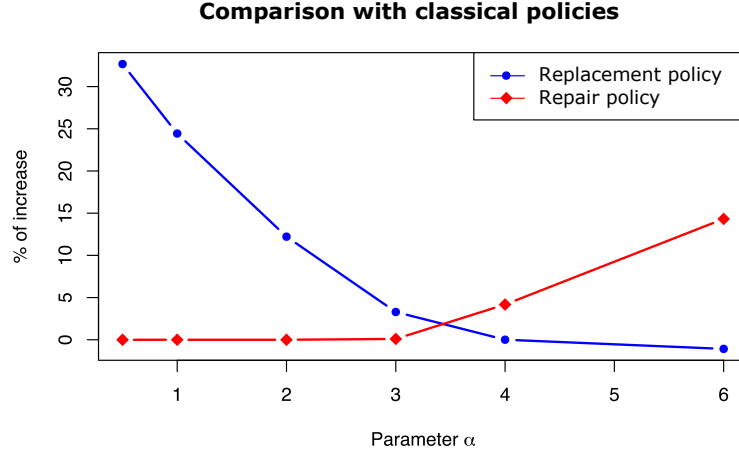


Figure 6: Cost comparison of (p,M,s)-policy with pure preventive replacement ($s = 0$) and pure repair policies ($s = \varphi(L)$). The efficiency of preventive actions decreases by increasing values of α while $\beta = 5$ is fixed.

perfect preventive one. In this case, it is more than 15% more efficient than an imperfect preventive policy. In between, an effective optimal balance is obtained. In conclusion, the introduction of a new decision variable gives a powerful adaptive policy.

5.3 Decision variable for replacement or repair

Now let focus on the meaning of the decision variable s . At each inspection and for a current degradation level $M \leq x < L$, $\varphi(x)$ is the probability that the degradation level after imperfect repair would be still greater than M . The decision about replacement or repair is made by comparing $\varphi(x)$ and s . Consequently, the degradation level ω such that $\varphi(\omega) = s$ plays a specific role. As φ is an increasing bijective function of x , $\varphi(x) \leq s$ (respectively $\varphi(x) > s$) is equivalent to $x \leq \omega$ (respectively $x > \omega$), see Figure 7. The limit threshold s on probability $\varphi(x)$ can be considered as a limit threshold ω on degradation level x . Consequently, it is possible to consider either s or the threshold on degradation level ω as the third decision variable.

Here is one reason why the decision variable s is chosen as a decision variable in this study rather than ω . Consider a “nominal” configuration for the system, which is characterized by efficient repair property. In the following, the tilde symbol will be used to indicate the parameters

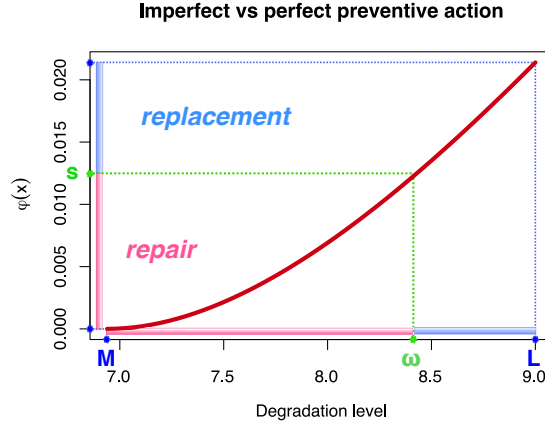


Figure 7: Decision variable s and associated degradation level threshold.

and variables corresponding to the nominal configuration. The nominal shape parameter α is taken equal to $\tilde{\alpha} = 0.5$ which means that $\mathbb{E}[\delta] = 0.09$. The decision variables are optimized for this nominal configuration. The obtained values are \tilde{p}^* , \tilde{M}^* and \tilde{s}^* . Additionally, $\tilde{\omega}^*$ denotes the value of $\tilde{\varphi}^{-1}(\tilde{s}^*)$ where $\tilde{\varphi}$ is the function φ with nominal beta distribution parameters. Now, suppose that the repair efficiency decreases, i.e. α increases. Assume that it is not possible nor suitable to perform new optimizations. In that case, only the function φ can be updated with the new efficiency parameter. Two options can be considered:

- the first one is referred as “ s constant” which means that s is fixed and remains equal to \tilde{s}^* . As the efficiency changes, so do φ and the limit degradation level $\omega = \varphi^{-1}(\tilde{s}^*)$;
- the second option is called “ ω ” constant”. In that case, ω remains equal to its initial value $\tilde{\omega}^*$ and the decision variable $s = \varphi(\tilde{\omega}^*)$ is updated with φ function.

The robustness of the maintenance policy in case of a decrease in repair efficiency is evaluated for each option. We first assess the maintenance costs $EC(\tilde{p}^*, \tilde{M}^*, \tilde{s}^*)$ and $EC(\tilde{p}^*, \tilde{M}^*, \varphi(\tilde{\omega}^*))$ associated respectively to the cases “ s constant” and “ ω ” constant”. The relative difference with the cost value $EC(p^*, M^*, s^*)$ which is optimized with the exact value of repair efficiency is calculated. The results are depicted on Figure 8. Recall that in the example considered, the fixed decision variables \tilde{s}^* and $\tilde{\omega}^*$ are optimal for efficient repair and lead to an imperfect preventive strategy. As α increases, the optimal strategy remains first a repair (imperfect) strategy. Then it shifts to a replacement (perfect) strategy as α becomes greater than 3. Figure 8 shows that after this shift, the option “ s constant” gives significantly better performance. Different similar tests lead

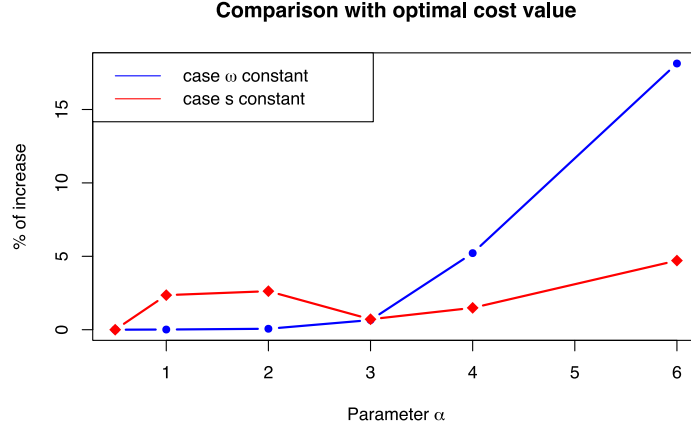


Figure 8: Cost variations for fixed decision variables for a decreasing efficiency of imperfect actions. Decision variables have been optimized for $\alpha = 0.5$ and results are given as a percentage of the corresponding optimal cost. Diamonds are for “ s constant” and bullets for “ ω constant”.

to the recommendation to keep the initial optimum of s rather than ω in case of a change in repair efficiency if no new cost optimization is to be considered. This may result in slightly lower performance if efficiency repair remains close to the nominal one. However, it is more broadly beneficial when changes in efficiency lead to an evolution of the optimal strategy (e.g., from preventive repair to preventive replacement). Finally, considering s as a decision variable is more robust than considering ω .

6 Sensitivity analysis

This section deals with the influence of some problem’s parameters on the optimal configurations of the maintenance decision rule. Parameters with the main influence on the balance between repair and corrective replacement are investigated.

6.1 Sensitivity to unit maintenance costs

Unit costs play an essential role in determining the optimal configuration. The evolution of the optimal (p, M, s) -policy is studied in this section when there are modifications in repair and preven-

C_r	p^*	M^*	s^*	ω^*	EC^*
2.0	0.0528	7.25	1.4e-03	9	0.66
4.0	0.0528	7.25	1.4e-03	9	0.91
4.5	0.0626	7.25	1.1e-03	8.9	0.98
5.0	0.0697	7.25	9.0e-06	7.79	1.021
5.5	0.0604	7.68	6.0e-18	7.68	1.024
7.0	0.0607	7.69	5.0e-17	7.69	1.024

Table 1: Optimal configurations for different values of preventive repair unit cost. Imperfect maintenance is such that $\alpha = 2$ and $\beta = 5$. Unit costs are respectively $C_i = 0.2$, $C_{pr} = 7$, $C_{add} = 7$, $C_{cr} = 10$, $C_d = 4$.

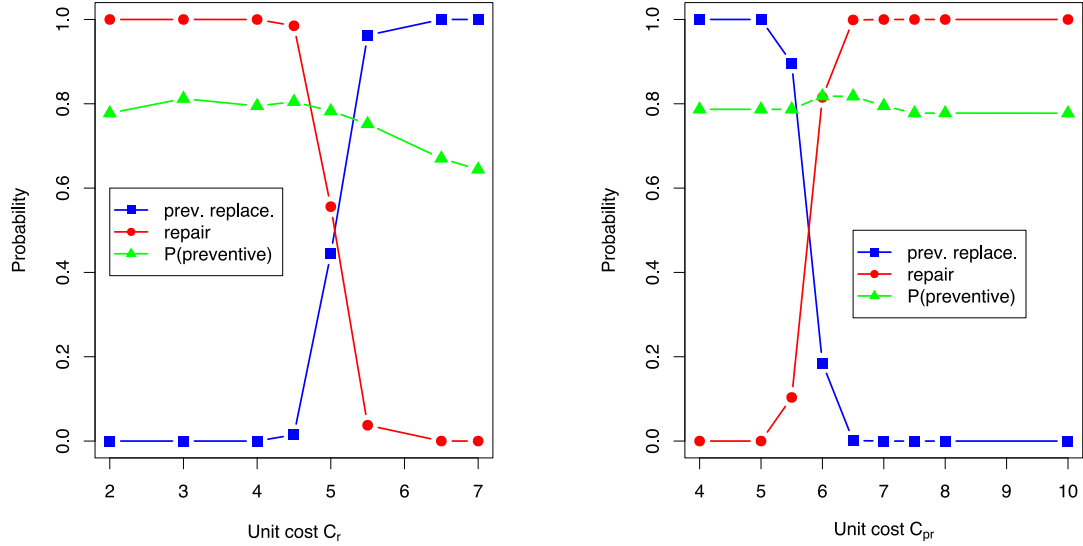
tive replacement unit costs. As an illustration, the case with $\alpha = 2$ and $\beta = 5$, i.e. $\mathbb{E}[\delta] \simeq 0.286$, is proposed for repair efficiency. The cost configuration is also the same as in (5.1).

First, let address the influence of repair unit cost C_r . We consider it varies from 2 to 7. The value $C_r = 2$ corresponds to a low cost for imperfect maintenance while the case $C_r = 7$ is a limit case where repair is as costly as preventive replacement. As a consequence and given the value of $C_{add} = 7$, C_{r+} increases from 9 to 14. Thus, the cost of inevitable replacement after an inefficient repair, C_{r+} can be almost the same as a corrective replacement or even more costly. Table 1 shows the evolution of decision variables and optimal cost rate as C_r increases.

With the increase in C_r or, in other words, as corrective actions become economically less disadvantageous, the value of p^* increases. Decision variable M^* remains almost constant for smaller values of C_r and increases for higher values. The values of s^* are such that $\omega^* = \varphi^{-1}(s^*)$ decreases from L to M^* as C_r increases. It means that the maintenance policy is prioritizing imperfect preventive actions for low values of C_r and preventive replacements for high values of C_r .

Figure 9a illustrates the evolution of the probabilities related to preventive actions. Three curves are plotted. The one with triangles highlights the probability that a cycle ends with preventive action. It decreases as C_r increases. The two other curves are related to the probabilities of having a repair (resp. a replacement) given that the cycle ends with a preventive maintenance action. A

sudden switching is clearly visible between $C_r = 4$ and $C_r = 6.5$. It traduces a high sensitivity to repair unit cost in this area.



(a) C_{pr} is fixed and equal 7 while C_r is changing

(b) C_r is fixed and equal 4 while C_{pr} is changing

Figure 9: Evolution of probabilities of repair and preventive replacement knowing maintenance action is preventive as functions of unit costs, for $\alpha = 2$ and $\beta = 5$ with $\mu = \lambda = 1$, $L = 9$, $C_i = 0.2$, $C_{add} = 7$, $C_{cr} = 10$ and $C_d = 4$.

Let now consider the influence of preventive replacement unit cost C_{pr} . Let consider C_{pr} takes values in the range of 4, which is the value of C_r up to 10, which is the value of C_c . Numerical results are given on Table 2 and Figure 9b. Note that p^* varies slightly whereas M^* remains constant. The value of s^* also increases due to the fact that preventive actions are only replacements for low values of C_{pr} and and only repairs for high values of C_{pr} .

As depicted with the evolution of C_r , the proposed curves represent the evolution of the probability that a cycle ends with preventive action and, respectively, probabilities of repair and preventive replacement knowing the maintenance action is a preventive one. In case of low cost for preventive replacement, i.e., with C_{pr} close to C_r , repair is avoided. When C_{pr} becomes higher than 5, the choice for preventive actions switches from replacement to repair. The policy's sensitivity to variations of C_{pr} is very low for small or high values of perfect preventive cost. However, it

C_{pr}	p^*	M^*	s^*	ω^*	EC^*
4	0.038	7.30	4.4e-17	7.30	0.73
5	0.044	7.30	1.3e-12	7.33	0.83
5.5	0.045	7.30	2.6e-17	7.30	0.88
6	0.048	7.30	8.0e-05	8.20	0.91
6.5	0.054	7.30	1.2e-03	8.99	0.91
8	0.054	7.30	1.2e-03	9.00	0.91
10	0.054	7.30	1.2e-03	9.00	0.91

Table 2: Optimal configurations for different values of preventive replacement unit cost. Imperfect maintenance is such that $\alpha = 2$ and $\beta = 5$. Unit costs are respectively $C_i = 0.2$, $C_r = 4$, $C_{\text{add}} = 7$, $C_{\text{cr}} = 10$, $C_d = 4$.

increases suddenly between these two areas, i.e., between 5 and 7 for the considered example. In the case of preventive maintenance, the balance between repair and replacement decisions must be tuned accurately.

6.2 Sensitivity to maintenance efficiency

This section focuses on analyzing the optimal cost and its related decision variables as a function of repair efficiency. To this end, Table 3 and Figure 10 show the evolution of the optimal maintenance configuration as repair efficiency decreases. As in previous sections, the loss of repairs efficiency is related to the increase of $\mathbb{E}(\delta)$. The first shape parameter α varies from 0.5 to 6 while the second shape parameter β is fixed equal to 5. The initial case is associated to the most “efficient” repair actions, with $\mathbb{E}(\delta) \simeq 0.09$. The last one is related to the less efficient ones with $\mathbb{E}(\delta) \simeq 0.55$. Density and quantile functions of corresponding beta distributions are depicted in Figure 1. The unit costs are the same as in (5.1).

The results in Table 3 show that the optimal maintenance cost rate increases as the maintenance efficiency decreases. This increase is due to the interaction of two effects. On one hand, the mean degradation level increases with $\mathbb{E}(\delta)$ in case of a repair. As a consequence, the mean of the time before the next maintenance action decreases. On the other hand, low maintenance efficiency leads

α	0.5	1	2	3	3.6	4	5	6
$\mathbb{E}(\delta)$	0.091	0.167	0.286	0.375	0.419	0.444	0.5	0.545
EC^*	0.77	0.82	0.91	0.99	1.02	1.02	1.02	1.02

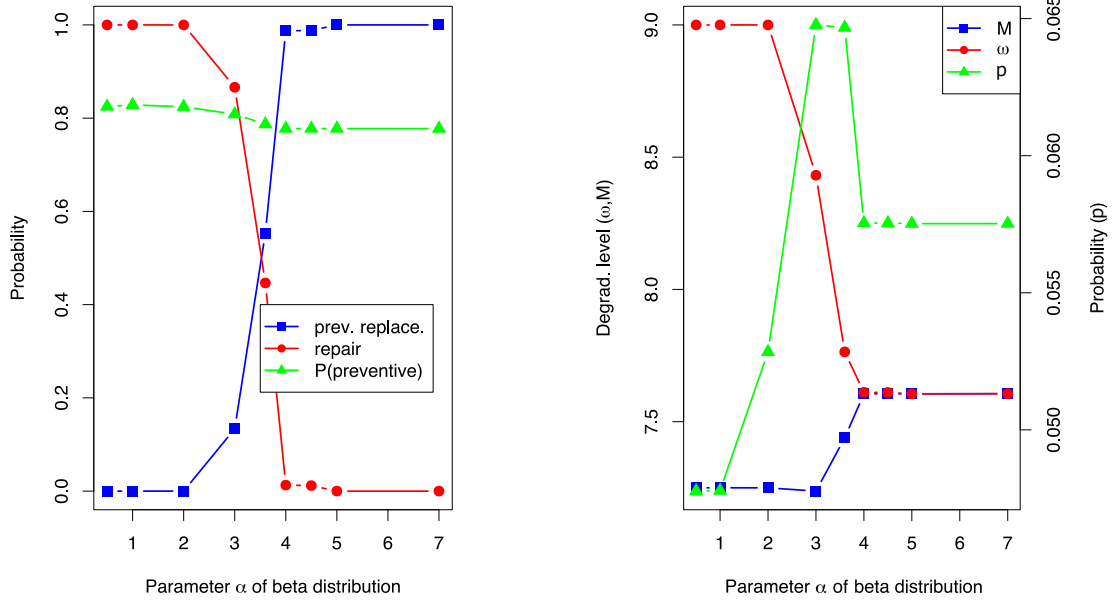
Table 3: Optimal maintenance cost rate for increasing value of α and fixed $\beta = 5$. The degradation parameters are $\mu = \lambda = 1$, and the failure level is $L = 9$.

to a preference for costly replacements. The switching between repair and replacement choice in case of preventive maintenance actions is visible in Figure 10, when α is between 2 and 5. The balances between unit cost strongly influence the values of optimal decision variables. For unit costs considered in Figure 10a, the probability that a cycle will end with preventive action is close to 0.8; see the line with triangles. The proportion of about 80% preventive actions and 20% corrective actions corresponds to the optimal balance for the given unit costs. It remains stable, albeit slightly decreasing from 0.82 to 0.78. The probability of having a scenario with an unsuccessful repair immediately followed by a replacement remains negligible. This specific case is very costly and, therefore, must be avoided by the optimal policy.

Figure 10b shows the evolution of optimal decision variables. The value $\omega^* = \varphi^{-1}(s^*)$ is provided instead of s^* for a more intuitive interpretation of the impact of the optimal maintenance decision variable s^* . When ω^* is close to M^* , preventive maintenance is a replacement. When ω^* is close to L , preventive actions are repairs, i.e., imperfect actions. The proposed (p,M,s)-policy behaves as an imperfect preventive policy for small values of α , namely from 0.5 to 2. Then it changes gradually to become a perfect preventive policy when α is greater than 5. In that case, an increase of α does not have any influence anymore because the probability of repair action is null.

7 Conclusion

The problem of imperfect maintenance in the context of deteriorating systems has already been addressed in different ways. However, this issue is still widely open. This paper aims to focus on imperfect maintenance efficiency and propose an adapted maintenance modeling framework under an aperiodic inspection and maintenance scheme. Inverse Gaussian process is considered for



(a) Probability of preventive maintenance, (b) Decision variables p^* , M^* and s^* (s^* is replaced by $\omega^* = \varphi^{-1}(s^*)$ for better readability).

Figure 10: Evolution of optimal configuration as a function of maintenance efficiency for increasing value of α and with fixed $\beta = 5$. The degradation parameters are $\mu = \lambda = 1$, and the failure level is $L = 9$.

degradation modeling, and a random improvement factor model describes the effect of imperfect repair actions on the degradation level. To characterize the randomness in the effect of imperfect repair, we assumed that the random improvement factor follows a beta distribution. Meanwhile, the efficiency of maintenance actions is defined by parameters of beta distribution that provide a wide range of configurations. A predictive condition-based maintenance strategy is developed and assessed, allowing adaptation of maintenance decision-making to imperfect maintenance efficiency. The proposed decision rule involves entirely perfect and imperfect policies, which are particular cases. The balance between imperfect repairs and preventive replacement is made by introducing a new decision variable. With this definition, classical cases can be obtained for specific values of decision variables. The closed-form of the cost function is assessed based on semi-regenerative properties. Numerical experiments show the flexibility of the proposed maintenance modeling framework and its ability to adapt to imperfect efficiency levels.

Several aspects warrant further investigation. A future point of interest is to consider systems with changing maintenance efficiency. In different real-world cases, maintenance efficiency decreases with time or depending on the number of past repairs. Further study will be devoted to deriving a dynamic maintenance policy that can adapt automatically to evolutions of repair efficiency. In this context, we analyze the interest of a limit on the number of successive imperfect maintenances before a replacement.

In our policy, we assume that inspections lead to the exact knowledge of the system degradation level. This hypothesis is not compatible with all cases of application. It will be more complex to account for imperfect condition estimation because of consequences on the underlying stochastic process. It needs to describe degradation phenomenon and noise measurement jointly, as attempted, e.g., in [12] with two Wiener processes in case of deteriorating sensors. In this context, the question of estimation of the repair efficiency has to be explicitly addressed. The Bayesian framework should be of interest to take into account the information collected online.

In addition, other research may be useful to address other assumptions made in this work and extend its applicability. For example, it is of interest to consider the case of non-homogeneous degradation. Two specific issues arise in that case: (i) the use of the semi-regenerative technique is not straightforward, and work is needed to derive the theoretical study for long-run cost rate calculation, and (ii) modeling of repair efficiency must be modified. In this context, maintenance efficiency can be described in terms of degradation reduction as well as age or time reduction, e.g. see [9, 16]. Improvements factors on degradation level and time could be jointly considered.

Appendix A

Hybrid numerical method for integral equation

To assess the stationary law, we have to find $B(y)$ which is the solution of the following integral equation:

$$B(y) = K(y, 0) + \int_0^M K(y, x)B(x)dx, \quad (\text{A.1})$$

where function $K(y, x)$ has the following structure:

$$K(y, x) = A_2(x) + f_{\tau_p(x)}(y - x) \cdot \mathbb{I}_{\{x < y\}}.$$

Due to the structure of $K(y, x)$, equation A.1 is a mix between two kinds of integral equations of the second kind: Fredholm equation (integration from 0 to M) and Volterra equation (integration from 0 to y). Each of them has an associated numerical scheme, see [20]. To solve it numerically, we use the numerical scheme described hereafter.

Let discretize the interval $[0, M]$ with $N + 1$ points. The step is $h = M/N$. Notations are as follows for $i = 1, \dots, N + 1$:

$$\begin{aligned} x_i &= y_i = (i - 1)h, \\ B(x_i) &= B(y_i) = B_i, \\ K_{ij} &= K(y_j, x_i). \end{aligned}$$

($x_1 = y_1 = 0$ and $x_{N+1} = y_{N+1} = M$).

As for solving Volterra integral equation, Equation (A.1) is first rewritten as:

$$\left(1 - \int_0^M K(y, x) dx\right) B(y) = K(y, 0) + \int_0^M K(y, x) (B(x) - B(y)) dx. \quad (\text{A.2})$$

Then the integral on the right-hand side is approximated with a simple trapezoidal scheme:

$$\begin{aligned} \int_0^M K(y, x) (B(x) - B(y)) dx &\approx \frac{h}{2} K(y, x_1) (B(x_1) - B(y)) + h \sum_{j=2}^N K(y, x_j) (B(x_j) - B(y)) \\ &\quad + \frac{h}{2} K(y, x_{N+1}) (B(x_{N+1}) - B(y)). \end{aligned}$$

From equation (A.2) and taking $y = y_i$, $i = 1, \dots, N + 1$, it comes:

$$\begin{aligned} \left(1 - \int_0^M \bar{K}(y_i, x) dx\right) B_i &= \\ \bar{K}_{i1} + \frac{h}{2} \bar{K}_{i1} (B_1 - B_i) + h \sum_{j=2}^N \bar{K}_{ij} (B_j - B_i) + \frac{h}{2} \bar{K}_{i, (N+1)} (B_{N+1} - B_i) \end{aligned} \quad (\text{A.3})$$

then

$$\begin{aligned} \left(1 - \int_0^M \bar{K}(y_i, x) dx - h \left\{ \frac{1}{2} \bar{K}_{i1} + \sum_{j=2}^N \bar{K}_{ij} + \frac{1}{2} \bar{K}_{i, (N+1)} \right\}\right) B_i &= \\ \bar{K}_{i1} + h \left\{ \frac{1}{2} \bar{K}_{i1} B_1 + \sum_{j=2}^N \bar{K}_{ij} B_j + \frac{1}{2} \bar{K}_{i, (N+1)} B_{N+1} \right\}. \end{aligned} \quad (\text{A.4})$$

Let now denote

- B the vector of $B_j, j = 0, \dots, N + 1$

- K the matrix $(K_{ij})_{1 \leq i, j \leq N+1}$,
- $K^{(j)}$ the j -th column of K
- $K_{(i)}$ the i -th row of K
- V_{co} the vector of length $N + 1$ such that $V_{co} = (1/2, 1, \dots, 1, 1/2)^T$,
- M_{co} the diagonal matrix of size $N + 1$ with diagonal V_{co} .

With this consideration, equations (A.4) for $i = 1, \dots, N + 1$ can be written as:

$$\left(1 - \int_0^M K(y_i, x) dx - hK_{(i)}V_{co}\right)B_i = K_{i1} + h(KM_{co})_{(i)}B. \quad (\text{A.5})$$

Therefore, by considering the system of all the equations (A.5), B can be obtained as the solution of the linear system

$$(I_{N+1} - D - hKM_{co})B = K^{(1)},$$

where

- I_{N+1} is the identity matrix of size $N + 1$,
- D is a diagonal matrix with $\int_0^M K(y_i, x) dx - hK_{(i)}V_{co}$ in i -th position of the diagonal,
- the values of integrals $\int_0^M K(y_i, x) dx$ are obtained directly (i.e. with a classical integration function from software library).

Appendix B

Cost calculation with log barrier for numerical optimization

In order to find the decision variables (s^*, p^*, M^*) , the following nonlinear optimization problem with inequality constraints must be solved numerically:

$$\min_{s, p, M} \text{EC}(s, p, M)$$

$$\text{with } 0 \leq s \leq \varphi(L)$$

$$0 \leq p \leq 1$$

$$0 \leq M \leq L$$

An implementation of the Nelder-Mead algorithm with R language is considered for the numerical optimization process. Equivalent functions are available in most common languages (e.g. R, Python, Matlab, ...). Such an implementation requires providing a function that allows the evaluation of the maintenance cost to be minimized for a given value of decision variables. To take into account the inequality constraints, a penalization method with a logarithmic barrier is used. The optimization problem with constraints is transformed into an unconstrained penalized problem. Let \widetilde{EC} denote the log barrier penalized function. It is defined by:

$$\widetilde{EC}(s, p, M) = EC(s, p, M) + \mu \left(\ln((-s)^+) + \ln((s - \varphi(L))^+) + \ln((-p)^+) + \ln((s - 1)^+) + \ln((-M)^+) + \ln((M - L)^+) \right)$$

where μ is the barrier parameter and x^+ is the positive part of x (i.e. $x^+ = x$ if $x > 0$ and $x^+ = 0$ if $x < 0$). In this paper, the weight parameter μ is chosen equal to 10^6 . The successive steps for cost calculation are as follows:

For given values of decision variables s , p and M :

- evaluation of the stationary law (Eq. 4.8 to 4.10)
- evaluation of the maintenance cost $EC(s, p, M)$ (expectation w.r.t. stationary law as given in section 4.2)
- evaluation of the penalized function \widetilde{EC}

Elementary trapezoidal algorithms like Simpson's rule are considered for integrals calculation at step 2.

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