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► **To cite this version:**

Franck Corset, Mitra Fouladirad, Christian Paroissin, Emmanuel Remy. A parametric lifetime model for a multicomponents system with spatial interactions. *Applied Stochastic Models in Business and Industry*, 2018, 35 (2), pp.221-233. 10.1002/asmb.2449 . hal-02290382

**HAL Id: hal-02290382**

**<https://utt.hal.science/hal-02290382v1>**

Submitted on 24 May 2023

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# A parametric lifetime model for a multicomponents system with spatial interactions

F. Corset<sup>a</sup>, M. Fouladirad<sup>b\*</sup>, C. Paroissin<sup>c</sup>, E. Remy<sup>d</sup>

**Summary:** In this paper, we consider a system made of  $n$  components displayed on a structure (e.g. a steel plate). We define a parametric model for the hazard function which includes covariates and spatial interaction between components. The state (non-failed or failed) of each component is observed at some inspection times. From these data, we consider the problem of model parameter estimation. To achieve this, we suggest to use the SEM algorithm based on a pseudo-likelihood function. A definition for the time-to-failure of the system is given, generalizing the classical cases. A study based on numerical simulations is provided to illustrate the proposed approach.

**Keywords:** Parameter estimation, Pseudo-likelihood, SEM algorithm, Spatial physical interaction

## 1. INTRODUCTION

1 The failure of industrial systems can cause irreversible damages, economic and human  
2 losses, and also bring an unrecoverable impact on the environment. In order to avoid such  
3 undesirable events, the system lifetime has to be studied, which implies to point out the  
4 interactions between the components constituting the system and to propose a tractable

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5 model to describe and predict the system failure. A suitable model must be complex enough  
6 to take into account most of the different features of the system and simple enough to  
7 be used by practitioners in the field. One of the issues in system lifetime prediction is to  
8 properly take into account the system behaviour which means on the one hand the impact  
9 of the environmental conditions, and on the other hand the spatial interaction between the  
10 components.

11 For complex equipment, it is convenient to define the system state by a binary random  
12 variable. As long as the system is operational, its health indicator is equal to one and as soon  
13 as a failure occurs the health indicator is equal to zero. This model can ignore the complexity  
14 of the system and makes some potentially unrealistic, but necessary, simplifications on the  
15 system lifetime distribution. However, in our context, it allows to take advantage of existing  
16 results on binary data analysis with covariates and spatio-temporal models. Indeed, during  
17 the last decades, a great attention has been given to this subject. In Diggle et al. [9] and  
18 Gumpertz et al. [14], a marginal logistic regression model is used to characterize the impact  
19 of covariates on binary data. These developments are based on the notion of neighbourhood  
20 which is defined as a group of components related due to their spatial interaction. After the  
21 definition of what is called a neighbourhood, Besag [2, 3] proposed an auto-logistic model for  
22 spatial binary data, assuming a simple dependence on surrounding neighbours which may  
23 interact with each other. The interest of this model is largely pointed out in the literature for  
24 instance refer to [1, 12, 16, 20, 21, 22, 25]. They all brought out its simplicity and tractability  
25 in presence of data. This model is extended by Gumpertz et al. and Huffer et al. [13, 17] in  
26 order to integrate covariates. Afterwards, Zhu et al. [26] give another extension of this model  
27 in order to take into account the temporal dependency. Later Caragea and Kaiser [5] point out  
28 that these models are not suitable in the case of a neighbourhood with a large number of failed  
29 components and propose a centered model in which the failure probability of each component  
30 increases or decreases around the mean behaviour of the model describing the whole system.

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31 Each of these models is used in the framework of parameter inference. The natural choice  
32 of estimation technique for these models is the maximum likelihood method. However, for  
33 the auto-logistic model with spatially correlated variables, the hypothesis of independence of  
34 the observations is no longer valid. To overcome this difficulty, Besag [4] proposes to use the  
35 pseudo-likelihood which mimics the likelihood function under the hypothesis of independence  
36 of the observations. Its maximization is in general easy to implement and results have good  
37 statistical properties when the considered neighbourhood is not very large [24, 11]. A Monte  
38 Carlo maximum likelihood method is proposed by Geyer and Thompson [11]. Alternatively,  
39 Zhu et al. [26] use maximum pseudo-likelihood for parameter estimation and develop a  
40 Markov Chain Monte Carlo (MCMC) algorithm for predicting future observations. Li [18]  
41 studies the existence of the joint distribution for a two-step centered auto-logistic causal  
42 model by considering the process as a Markov chain Markov field [15]. However, since the  
43 joint distribution is untractable, he proposes the expectation-maximization pseudo-likelihood  
44 estimation method and studies its performance.

45 In this paper, the model is inspired by a real industrial system installed in electric power  
46 plants of EDF, the main French electric utility company. This system may be assimilated to  
47 a large plate vertically maintained by screws regularly distributed all over the surface of the  
48 plate. The failure of one screw leads to an increase of mechanical stress on the other screws  
49 surrounding the one that has failed, consequently increasing their own failure probability.  
50 When a fixed number of failed screws is reached, the system is considered to be failed in turn  
51 (the plate is no more sufficiently maintained and may fall down). In addition to this spatial  
52 physical interaction between the screws, environmental conditions and material properties  
53 may also impact the reliability of the screws, and thus the lifetime of the whole system.

54 Since the state (failed/non-failed) of the components is known thanks to inspections, the  
55 available data is binary but censored by interval and the main objective is to model both the  
56 impact of the covariates and the interactive neighbourhood on the components' lifetime. In

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57 reliability analysis, the impact of environmental conditions on the systems lifetime is often  
58 modelled via covariates through proportional hazards or additive hazards models [7, 19, 23].  
59 Therefore, in difference with the models studied in the literature and previously presented  
60 in this section, we have here the constraint of using the hazard rate and we shall model the  
61 impact of the spatial interaction on the failure rate of components. In this framework, we  
62 integrate the censored data in the statistical inference and suggest a parameter estimation  
63 procedure. Similarly to results previously exposed, a pseudo-maximum likelihood in the  
64 framework of Monte Carlo simulations is proposed.

65 The remainder of this paper is organized as follows. In Section 2, the problem statement  
66 and the survival model are described. Section 3 is devoted to parameter estimation and  
67 statistical studies. The system failure and time-to-failure distribution are detailed in Section  
68 4. In Section 5, a numerical example is considered to illustrate the proposed methodology.  
69 The paper ends with a conclusion and some prospects.

## 2. MODEL DESCRIPTION

70 We consider a system made of  $n$  components displayed on a structure, say, for instance, a  
71 set of screws on a steel plate or on a steel cylinder, see Figure 1 giving an illustration of  
72 such a system. We denote by  $V = \{1, \dots, n\}$  the set of all components. Let  $T_1, \dots, T_n$  be the  
73 components' lifetimes. From these lifetimes, we can construct  $n$  binary stochastic processes  
74 which switch from 0 to 1 after failure of the corresponding component: for any  $i \in V$  and for  
75 any  $t \geq 0$ ,  $Y_i(t) = \mathbf{1}_{T_i \leq t}$ . The failure of a component may lead to a higher mechanical stress  
76 on the components in its neighbourhood. Hence, we aim at determining if each failure of a  
77 component has an impact on the neighbouring components. This impact is assumed to lead  
78 to an increase of the hazard function of these components. Let  $B_i$  be the neighbourhood  
79 of the  $i$ -th component. We assume that symmetry about the neighbourhood holds, in the

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80 sense that if  $i_1 \in B_{i_2}$  then  $i_2 \in B_{i_1}$ . This combination of components and neighbours can  
 81 be viewed as a non-oriented graph where vertices are the components and where the edge  
 82 between vertices  $i_1$  and  $i_2$  exists if  $i_1 \in B_{i_2}$ .

83 As the main example all along this paper, we assume that  $n$  is a square number and that  
 84 the components are located on a regular  $\sqrt{n} \times \sqrt{n}$  grid (of course, more complex topologies  
 85 could be considered). For such an example, natural definitions of neighbourhood are the  
 86 following ones. The first one is the case where the neighbourhood can be defined as the four  
 87 cardinal points (up to the edge effect): in such a case, a component will have between 2 and 4  
 88 neighbours (see Figure 1 on the left). A second example is the case where the neighbourhood  
 89 can be defined as the eight cardinal points (also up to the edge effect): in such a case, a  
 90 component will have between 3 and 8 neighbours (see Figure 1 on the right). Of course,  
 91 other neighbourhoods could be defined and also other system structures could be considered  
 92 (e.g. a cylinder). Note that sphericalness of the neighbourhood is not mandatory and, for  
 93 physical reasons, such assumption can be removed.

94 [Figure 1 about here.]

The lifetime distributions of the components will be defined through their hazard functions  
 by considering that covariates and spatial interactions act multiplicatively on them. Let  $\mathcal{H}_{i,t}$   
 be the increasing  $\sigma$ -field generated by all the binary stochastic processes defined above,  
 except the  $i$ -th one, and up to time  $t$ :

$$\mathcal{H}_{i,t} = \sigma(Y_j(s), s \leq t, j \in \{1, \dots, n\} \setminus \{i\}).$$

95 Since  $\mathcal{H}_{i,u} \subseteq \mathcal{H}_{i,t}$  for any  $u \leq t$ , therefore we have the following equality  $\lambda_i(u|\mathcal{H}_{i,u}) =$   
 96  $\lambda_i(u|\mathcal{H}_{i,t})$  for  $u \leq t$ .

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The hazard function of the  $i$ -th component, denoted  $\lambda_i$ , depends on the covariates and the state of its neighbours as follows:

$$\forall t \geq 0, \quad \lambda_i(t|\mathcal{H}_{i,t}) = \lambda(t) \exp \left( \alpha \sum_{j \in B_i} C_j(t) Y_j(t) + \boldsymbol{\beta}' \mathbf{Z}_i(t) \right), \quad (1)$$

where  $\lambda(\cdot)$  is the baseline hazard function,  $\alpha \in \mathbb{R}$ ,  $\boldsymbol{\beta} \in \mathbb{R}^p$  and where, for the  $i$ -th component,  $\mathbf{Z}_i(\cdot)$  is a vector of  $p$  time-dependent covariates (environmental conditions, such as temperature, or material properties, such as steel plate, etc.) and  $C_j(\cdot)$ ,  $j \in B_i$ , is a time-dependent covariate (it may be one component of the vector  $\mathbf{Z}_j(\cdot)$ , e.g. the mechanical stress). In fact, it could be sufficient to condition with respect to  $\{Y_j(t), j \in B_i\}$  instead of  $\mathcal{H}_{i,t}$ , but it will be more convenient to condition on  $\mathcal{H}_{i,t}$  when considering the cumulative hazard function and/or the conditional survival function. Indeed, the conditional survival function of the  $i$ -th component is given by:

$$\forall t \geq 0, \quad S_i(t|\mathcal{H}_{i,t}) = \exp \left( - \int_0^t \lambda_i(u|\mathcal{H}_{i,u}) du \right) \quad (2)$$

97 If  $\alpha = 0$ , it corresponds to the standard Cox model with time-dependent covariates.

To simplify, we assume that  $\lambda$  is the hazard function of the Weibull distribution with scale parameter  $a > 0$  and shape parameter  $b > 0$ :

$$\forall t \geq 0, \quad \lambda(t) = \frac{b}{a} \left( \frac{t}{a} \right)^{b-1}. \quad (3)$$

98 However, any other parametric model could be considered for the baseline hazard function.

99 We illustrate the effect of the components' physical interaction through two simulations on  
100 a square grid of  $50 \times 50$  components, without imposing special impact from the mechanical  
101 stress nor the other covariates (for all  $i \in V$ ,  $C_i(t) \equiv 1$  and  $\boldsymbol{\beta} = 0$ ). The outputs are plotted  
102 respectively in Figure 2 and Figure 3 (red points represent failed components). To perform  
103 the simulations, we have used Algorithm 1 described in Sub-section 3.2. The parameter of

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104 the Weibull hazard function was set as follows:  $a = 2$  and  $b = 2$ . Parameter  $\alpha$  has been set  
105 to 0 (no spatial interaction) and 2 (interaction). In both cases, the figures show the state  
106 of the system after 74, 274, 475 and 678 failures (for a total of 2500 components). We can  
107 observe that, on contrary to the model with physically independent components, clusters of  
108 failed components arise in the model with spatial interaction.

109 [Figure 2 about here.]

110 [Figure 3 about here.]

111 In the next section, we will propose an estimation procedure for the different parameters  
112 involved in the model described above. The approach is based on the SEM algorithm, a  
113 stochastic version of the EM algorithm proposed by Celeux et al. [6]. To this purpose, we  
114 need to simulate missing observations and thus simulation algorithms will be provided.

### 3. STATISTICAL INFERENCE

115 Assume that the system is not often inspected and the inspections are not necessarily  
116 periodical. At each control, all the components of the system are inspected. Consequently  
117 the inspection times are the same for all the components: we will denote by  $m$  the number  
118 of inspections and by  $\tau_1 < \dots < \tau_m$  the associated deterministic times. The available data  
119 at each inspection time is a binary information on whether each component is failed or  
120 not. Since the failures may occur between two successive controls, the data are interval-  
121 censored. Here, we neglect the possibility of non-detection of a failure during an inspection.  
122 For the time-dependent covariates  $C_1, \dots, C_n$  and  $\mathbf{Z}_1, \dots, \mathbf{Z}_n$ , it will be assumed that they  
123 are observed continuously, otherwise they will be constant piece-wise or linear piece-wise  
124 interpolated.



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125 In Sub-section 3.1, we provide a methodology to estimate the parameters of the model (we  
 126 consider a simplified parametric model). This approach requires to simulate missing data  
 127 and the useful simulation algorithms are detailed in Sub-section 3.2.

### 128 3.1. Parameter estimation

129 In order to proceed parameter estimation, first the expression of the likelihood function is  
 130 given. It is highlighted that it is numerically too complex to use this function to estimate  
 131 the model parameters. Hence, we suggest to rather use the SEM algorithm jointly with a  
 132 pseudo-likelihood function. Before giving details about the proposed approach, we introduce  
 133 some notations.

#### 134 3.1.1. Notation

We shall denote by  $\mathbf{Y}_{obs}$  the set of all the observations  $\{Y_i(\tau_j), i \in V, j \in \{1, \dots, m\}\}$ . Notice that  $\mathbf{Y}_{obs} \in \bigcup_{i=1}^m \mathcal{H}_{i, \tau_m}$ . Let  $\mathcal{N}_1$  be the set of all failed components at the last inspection time  $\tau_m$ . For a component  $i$ ,  $1 \leq i \leq n$ , let  $j_i = \min\{j; Y_i(\tau_j) = 1\} \in \{1, \dots, m\}$  be the inspection time such that the failure of component  $i$  is observed for the first time. Thus, for the  $i$ -th component, the failure occurs between  $\tau_{j_i-1}$  and  $\tau_{j_i}$ . For convenience, we denote by  $I_r$  the set of components failing between  $\tau_{r-1}$  and  $\tau_r$  for any  $r \in \{1, \dots, m\}$  (with the convention that  $\tau_0 = 0$ ):

$$I_r = \{i \in \mathcal{N}_1; j_i = r\} \quad \text{and} \quad n_r = |I_r|.$$

135 Hence, we have  $\mathcal{N}_1 = I_1 \cup \dots \cup I_m$ . Let  $I_{m+1} = V \setminus \mathcal{N}_1$  be the set of all components that  
 136 have survived until the last inspection time  $\tau_m$ . Let  $n_{m+1} = |I_{m+1}|$ . Clearly we have  $n =$   
 137  $n_1 + \dots + n_m + n_{m+1}$ . For any  $r \in \{1, \dots, m+1\}$ , let  $\mathcal{A}_r = \{T_i \in (\tau_{r-1}, \tau_r]; i \in I_r\}$ , with  
 138  $\tau_0 = 0$  and  $\tau_{m+1} = +\infty$ , be the set of events consisting of the components failed between  
 139  $\tau_{r-1}$  and  $\tau_r$ .

The likelihood function can be expressed as follows:

$$L(\alpha, \beta, a, b; \mathbf{Y}_{obs}; C_1, \dots, C_n, \mathbf{Z}_1, \dots, \mathbf{Z}_n) = \mathbb{P}[\mathcal{A}_1] \mathbb{P}[\mathcal{A}_2 | \mathcal{A}_1] \mathbb{P}[\mathcal{A}_3 | \mathcal{A}_1 \cap \mathcal{A}_2] \\ \times \dots \times \mathbb{P}[\mathcal{A}_m | \mathcal{A}_1 \cap \dots \cap \mathcal{A}_{m-1}] \mathbb{P}[\mathcal{A}_{m+1} | \mathcal{A}_1 \cap \dots \cap \mathcal{A}_m].$$

Let us compute  $\mathbb{P}[\mathcal{A}_1]$ :

$$\mathbb{P}[\mathcal{A}_1] = \sum_{\sigma_1 \in \Sigma_1} \mathbb{P}[0 < T_{\sigma_1(i_1)} < \dots < T_{\sigma_1(i_{n_1})} < \tau_1]$$

where  $\Sigma_1$  is the set of all permutations of elements  $\{i_1, \dots, i_{n_1}\}$  in  $I_1$ . We shall recall that  $|\Sigma_1| = n_1!$ . Next step consists in assessing all these  $n_1!$  probabilities:

$$\mathbb{P}[0 < T_{\sigma_1(i_1)} < \dots < T_{\sigma_1(i_{n_1})} < \tau_1] \\ = \int_0^{\tau_1} \mathbb{P}[0 < T_{\sigma_1(i_1)} < \dots < T_{\sigma_1(i_{n_1})} < \tau_1 | T_{\sigma_1(i_1)} = u_1] f_{T_{\sigma_1(i_1)}}(u_1) du_1 \\ = \int_0^{\tau_1} \mathbb{P}[u_1 < T_{\sigma_1(i_2)} < \dots < T_{\sigma_1(i_{n_1})} < \tau_1 | T_{\sigma_1(i_1)} = u_1] f_{T_{\sigma_1(i_1)}}(u_1) du_1$$

where

$$f_{T_{\sigma_1(i_1)}}(u_1) = \frac{b}{a} \left(\frac{u_1}{a}\right)^{b-1} \exp\left(-\left(\frac{u_1}{a}\right)^b\right).$$

Now we compute the inner probability. In the same way,

$$\mathbb{P}[u_1 < T_{\sigma_1(i_2)} < \dots < T_{\sigma_1(i_{n_1})} < \tau_1 | T_{\sigma_1(i_1)} = u_1] \\ = \int_{u_1}^{\tau_1} \mathbb{P}[u_2 < T_{\sigma_1(i_3)} < \dots < T_{\sigma_1(i_{n_1})} < \tau_1 | T_{\sigma_1(i_2)} = u_2] f_{T_{\sigma_1(i_2)} | T_{\sigma_1(i_1)}}(u_2 | u_1) du_2$$

---

where

$$f_{T_{\sigma_1(i_2)}|T_{\sigma_1(i_1)}}(u_2|u_1) = \lambda_{T_{\sigma_1(i_2)}|T_{\sigma_1(i_1)}}(u_2|u_1) \exp\left(-\int_0^{u_2} \lambda_{T_{\sigma_1(i_2)}|T_{\sigma_1(i_1)}}(v|u_1)dv\right)$$

with

$$\lambda_{T_{\sigma_1(i_2)}|T_{\sigma_1(i_1)}}(v|u_1) = \frac{b}{a} \left(\frac{v}{a}\right)^{b-1} \exp(\boldsymbol{\beta}' \mathbf{Z}_{i_2}(v))$$

if  $i_2 \notin B_{i_1}$ , and with

$$\lambda_{T_{\sigma_1(i_2)}|T_{\sigma_1(i_1)}}(v|u_1) = \begin{cases} \frac{b}{a} \left(\frac{v}{a}\right)^{b-1} \exp(\boldsymbol{\beta}' \mathbf{Z}_{i_2}(v)) & \text{for } v \leq u_1 \\ \frac{b}{a} \left(\frac{v}{a}\right)^{b-1} \exp(\alpha C_{i_1}(v) + \boldsymbol{\beta}' \mathbf{Z}_{i_2}(v)) & \text{for } v > u_1 \end{cases}$$

141 if  $i_2 \in B_{i_1}$ . Finally the probability  $\mathbb{P}[\mathcal{A}_1]$  appears to be a  $n_1$ -dimensional integral and thus  
 142 its contribution to the likelihood function is the sum of  $n_1!$  integrals in dimension  $n_1$ . It will  
 143 be similar for other contributions: for any  $j \in \{1, \dots, m+1\}$ ,  $\mathbb{P}(\mathcal{A}_j | \cap_{i=1}^{j-1} \mathcal{A}_i)$  is the sum of  
 144  $n_j!$  integrals in dimension  $n_j$ .

145 Consequently two problems may arise with the direct MLE method if some of the  $n_1, \dots,$   
 146  $n_{m+1}$  are too large: (1) at each step of the optimization algorithm, a numerical procedure has  
 147 to be used to compute an integral in possibly high dimension; (2) a combinatorial explosion  
 148 when considering the different permutations. To avoid such problems, an alternative method  
 149 is proposed. Before presenting this approach, we discuss some special cases. For instance, the  
 150 two previously mentioned problems will not arise if  $n_j \leq 2$  for all  $j \in \{1, \dots, m+1\}$ . This  
 151 may occur if  $m$  is close to  $n$  and if the components' lifetimes are uniformly distributed  
 152 between 0 and  $\tau_m$ . However, spatial physical interaction between components reinforces  
 153 clusters of failed components (as shown in the simulations, see Figure 3), thus this particular  
 154 case may arise essentially when there is no interaction between components.

To conclude the section, we give the likelihood function when components are physically independent (i.e., when  $\alpha = 0$ ) and when there is no effect of the covariates (i.e.,  $\boldsymbol{\beta} = 0$ ),

---

since we will need it later:

$$\log L(a, b; \mathbf{Y}_{obs}) = \sum_{r=1}^m n_r \log [S(\tau_{r-1}; a, b) - S(\tau_r; a, b)] + n_{m+1} \log [S(\tau_m; a, b)] \quad (4)$$

with

$$\forall t \geq 0, \quad S(t; a, b) = \exp \left( - \left( \frac{t}{a} \right)^b \right).$$

### 155 3.1.3. SEM algorithm and pseudo-likelihood

156 As seen above, the maximum likelihood estimator of the parameters cannot be computed  
 157 neither in a closed-form, nor numerically. Due to the censoring mechanism (failure times  
 158 are never observed exactly, only interval censored data are available), we have a problem  
 159 of missing observations. Indeed, hazard functions depend on binary stochastic processes  
 160  $Y_1, \dots, Y_n$  which are only observed at inspection times. A solution to this problem consists  
 161 in applying the SEM algorithm. This algorithm requires to simulate the missing data, which  
 162 are the times-to-failure of the components (this will be described in the next sub-section).  
 163 The SEM algorithm is described below.

164 **Step 0:** parameter initialization. In equation 1, set  $\hat{\alpha}^{(0)} = 0$  and  $\hat{\beta}^{(0)} = 0$  (no effect).  
 165 Parameters  $a$  and  $b$  can thus be estimated by maximizing the equation (4)(since  
 166 components are not interacting):  $\hat{a}^{(0)}$  and  $\hat{b}^{(0)}$ .

167 **Step  $k$ :** time-to-failures simulation and updating estimation.

- 168 1. simulate  $T_1, \dots, T_n$  using Algorithm 2 (see next Sub-section) and considering  
 169 parameters  $\hat{\alpha}^{(k-1)}, \hat{\beta}^{(k-1)}, \hat{a}^{(k-1)}$  and  $\hat{b}^{(k-1)}$ ;
- 170 2. update the estimation by maximizing numerically the log-likelihood function:  $\hat{\beta}^{(k)},$   
 171  $\hat{\alpha}^{(k)}, \hat{a}^{(k)}$  and  $\hat{b}^{(k)}$ .

---

The final estimator is then given by considering the ergodic average of the estimators. After  $K$  iterations of the algorithm, let:

$$\hat{\alpha} = \frac{1}{K} \sum_{k=1}^K \hat{\alpha}^{(k)}, \quad \hat{\beta} = \frac{1}{K} \sum_{k=1}^K \hat{\beta}^{(k)}, \quad \hat{a} = \frac{1}{K} \sum_{k=1}^K \hat{a}^{(k)} \quad \text{and} \quad \hat{b} = \frac{1}{K} \sum_{k=1}^K \hat{b}^{(k)}.$$

To finish the description of this approach, the expression of the likelihood function is provided. Algorithm 2 simulates in fact the random vector of order statistics  $T_{(1)}, \dots, T_{(n)}$  (notice that the algorithm simulates also the random vector of rank statistics and let us recall that the knowledge of these two random vectors is equivalent to the knowledge of  $T_1, \dots, T_n$ ). Hence, a likelihood function may be defined from such simulated data. However, a drawback of this approach is that it relies too strongly on the simulated missing observations and may fail (in fact, SEM is mainly used for unsupervised classification problem and, in such problems, only labels are not observed, so "less" data is not observed and less information is missing). For this reason, it is suggested to use the SEM algorithm based on a pseudo-likelihood function which is obtained by ignoring the dependency between observations. Pseudo-likelihood functions have been successfully considered in the literature, see, e.g., [8]. Here, we propose the following pseudo-likelihood function based on Equation (2):

$$\begin{aligned} pL(\alpha, \beta, a, b; \mathbf{Y}_{obs}, C_1, \dots, C_n, Z_1, \dots, Z_n, t_1, \dots, t_n) \\ &= \prod_{i \notin \mathcal{N}_1} S_i(\tau_m | \mathcal{H}_{i, \tau_m}) \prod_{i \in \mathcal{N}_1} [S_i(\tau_{j_i-1} | \mathcal{H}_{i, \tau_m}) - S_i(\tau_{j_i} | \mathcal{H}_{i, \tau_m})] \\ &= \prod_{i \in I_{m+1}} S_i(\tau_m | \mathcal{H}_{i, \tau_m}) \prod_{r=1}^m \prod_{i \in I_r} [S_i(\tau_{r-1} | \mathcal{H}_{i, \tau_m}) - S_i(\tau_r | \mathcal{H}_{i, \tau_m})] \end{aligned}$$

172 (a product over an empty set is considered to be equal to 1) where  $t_1, \dots, t_n$  are the missing  
 173 observation which are reconstructed by using Algorithm 2. In fact, the simulated lifetimes  
 174 are only used to determine the order of failure of each component.

---

175 **3.2. Simulation algorithms**

176 In this section, several simulation procedures are proposed. The first one will allow to simulate  
 177 data that will be used in the next section for the numerical illustrations of the methodology.  
 178 The second algorithm is used within the SEM algorithm.

179 *3.2.1. Data generation algorithm*

180 To generate lifetimes from the model described in Section 2, we can apply Algorithm 1. It  
 181 draws the lifetimes  $T_1, \dots, T_n$  and allows to construct the stochastic processes  $Y_1(\cdot), \dots, Y_n(\cdot)$ .  
 182 At least, to get data as considered in the previous Sub-section, we have to sample  
 183  $Y_1(\cdot), \dots, Y_n(\cdot)$  at times  $\tau_1, \dots, \tau_m$ . To achieve this, we can stop Algorithm 1 at step  $r$   
 184 such that  $T_{i_{r-1}} < \tau_m < T_{i_r}$  (it is not useful to simulate all the  $n$  lifetimes).

**Require:**  $n, a, b, \alpha, \beta, \mathbf{Z}_1(\cdot), \dots, \mathbf{Z}_n(\cdot)$

- 1:  $T \leftarrow \text{Vector}(\text{length} = n)$
- 2:  $\tilde{T} \leftarrow \text{Vector}(\text{length} = n)$
- 3: draw  $U_1, \dots, U_n$  i.i.d. from the uniform distribution over  $[0; 1]$
- 4: **{Step 1}**
- 5: set  $R \leftarrow \{1, \dots, n\}$  **{set of non-failed components}**
- 6: draw  $\tilde{T}_1, \dots, \tilde{T}_n$  by solving the following equation with respect to  $t_1, \dots, t_n$ :

$$\log(U_i) + \int_0^{t_i} \lambda(s) \exp(\beta' \mathbf{Z}_i(s)) ds = 0$$

- 7:  $i_1 \leftarrow \text{argmin}_{j \in R} \tilde{T}_j$  **{select the next component to fail}**
- 8:  $T_{i_1} \leftarrow \tilde{T}_{i_1}$  **{store the next failure times}**
- 9: **for**  $r \in \{2, \dots, n\}$  **do**
- 10: **{Step  $r$ }**
- 11:  $R \leftarrow R \setminus \{i_{r-1}\}$  **{update the set of non-failed components}**
- 12: **for**  $j \in B_{i_{r-1}} \cap R$  **do**
- 13: draw  $\tilde{T}_j$  by solving the following equation with respect to  $t_j$ :

$$\log(U_j) + \int_{T_{i_{r-1}}}^{t_j} \lambda(s) \exp \left( \alpha \sum_{k \in B_j} C_k(s) Y_k(s) + \beta' \mathbf{Z}_j(s) \right) ds = 0$$

- 14: **end for**
- 15:  $i_r \leftarrow \text{argmin}_{j \in R} \tilde{T}_j$  **{select the next component to fail}**
- 16:  $T_{i_r} \leftarrow \tilde{T}_{i_r}$  **{store the next failure times}**
- 17: **end for**
- 18: **return**  $T_1, \dots, T_n$

**Algorithm 1:** Model simulation

---

185 *3.2.2. Simulation algorithm using observation data*

186 In this paragraph, the general idea of the algorithm required to use the SEM algorithm  
187 is described. Roughly speaking, the idea is to simulate data by considering the successive  
188 observation intervals. Recall that, for any  $j \in \{1, \dots, m\}$ ,  $I_j$  is the set of components whose  
189 lifetimes are between  $\tau_{j-1}$  and  $\tau_j$  and  $I_{m+1}$  is the set of non-failed components at the last  
190 inspection time  $\tau_m$ . The procedure starts by simulating lifetimes for components in  $I_1$ , then  
191 in  $I_2$ , and so on. Algorithm 2 describes the suggested approach.

#### 4. SYSTEM TIME-TO-FAILURE

192 A definition of the system time-to-failure (system TTF, in short) can be based on a spatial  
193 extension of consecutive  $k$  out of  $n$  systems. In such a model, when components are displayed  
194 on a line (which appears to be a special case of our model) or on a circle, the system  
195 is assumed to be failed as soon as  $k$  consecutive components have failed. To achieve this  
196 generalization, several notations and random variables are introduced.

Let us recall that  $V = \{1, \dots, n\}$  denotes the set of all components. For any subset  $U \subseteq V$ ,  
set  $\mathcal{T}_U = \max_{i \in U}(T_i)$ . It corresponds to the time such that all components in  $U$  have failed.  
Next, for any  $k \in \{1, \dots, n\}$ , let :

$$\mathcal{U}_k = \{U \subseteq V; |U| = k \text{ and } U \text{ is connected}\}$$

be the set of all subsets made of  $k$  connected components. We recall that two components  
are connected if they are neighbours (in other words, if there exists an edge between these  
two components) and that a sub-graph is connected if there exists a path between any two  
components in this sub-graph. Imposing the connectivity of  $U$  is important due to the spatial  
interaction of the components. We could also impose a condition on the diameter of the sub-  
graph as follows (recall that the diameter of a graph is the longest shortest path between

---

**Require:**  $n, a, b, \alpha, \beta, \mathbf{Z}_1(\cdot), \dots, \mathbf{Z}_n(\cdot), \mathbf{Y}_{obs}$

- 1:  $T \leftarrow \text{Vector}(\text{length} = n)$
- 2:  $\tilde{T} \leftarrow \text{Vector}(\text{length} = n)$
- 3: draw  $U_1, \dots, U_n$  i.i.d. from the uniform distribution over  $[0; 1]$
- 4:  $\{\text{Interval } I_1\}$
- 5:  $R \leftarrow I_1$  {set of non-failed components in  $I_1$ }
- 6: **for**  $j \in I_1$  **do**
- 7:   draw  $\tilde{T}_j$  by solving the following equation with respect to  $t_j$ :

$$\log(U_j) + \int_0^{t_j} \lambda(s) \exp(\beta' \mathbf{Z}_j(s)) ds = 0$$

- 8: **end for**
- 9:  $h \leftarrow \text{argmin}_{j \in R} \tilde{T}_j$  {select the next component to fail}
- 10:  $T_h \leftarrow \tilde{T}_h$  {store the next failure times}
- 11: **if**  $n_1 > 1$  **then**
- 12:   **for**  $r \in \{2, \dots, n_1\}$  **do**
- 13:      $R \leftarrow R \setminus \{h\}$  {update the set of non-failed components in  $I_1$ }
- 14:     **for**  $j \in B_h \cap R$  **do**
- 15:       draw  $\tilde{T}_j$  by solving the following equation with respect to  $t_j$ :

$$\log(U_j) + \int_{T_h}^{t_j} \lambda(s) \exp \left( \alpha \sum_{k \in B_j} C_k(s) Y_k(s) + \beta' \mathbf{Z}_j(s) \right) ds = 0$$

- 16:   **end for**
- 17:    $h \leftarrow \text{argmin}_{j \in R} \tilde{T}_j$  {select the next component to fail}
- 18:    $T_h \leftarrow \tilde{T}_h$  {store the next failure times}
- 19: **end for**
- 20: **end if**
- 21: **for**  $i \in \{2, \dots, m\}$  **do**
- 22:   {Interval  $I_i$ }
- 23:    $R \leftarrow I_i$
- 24:   **if**  $n_i > 0$  **then**
- 25:     **for**  $j \in I_i$  **do**
- 26:       draw  $\tilde{T}_j$  by solving the following equation with respect to  $t_j$ :

$$\log(U_j) + \int_{\tau_{i-1}}^{t_j} \lambda(s) \exp \left( \alpha \sum_{k \in B_j} C_k(s) Y_k(s) + \beta' \mathbf{Z}_j(s) \right) ds = 0$$

- 27:   **end for**
- 28:    $h \leftarrow \text{argmin}_{j \in R} \tilde{T}_j$  {select the next component to fail}
- 29:    $T_h \leftarrow \tilde{T}_h$  {store the next failure times}
- 30:   **if**  $n_i > 1$  **then**
- 31:     **for**  $r \in \{2, \dots, n_i\}$  **do**
- 32:        $R \leftarrow R \setminus \{h\}$  {update the set of non-failed components in  $I_i$ }
- 33:       **for**  $j \in B_h \cap R$  **do**
- 34:         draw  $\tilde{T}_j$  by solving the following equation with respect to  $t_j$ :

$$\log(U_j) + \int_{T_h}^{t_j} \lambda(s) \exp \left( \alpha \sum_{k \in B_j} C_k(s) Y_k(s) + \beta' \mathbf{Z}_j(s) \right) ds = 0$$

- 35:   **end for**
- 36:    $h \leftarrow \text{argmin}_{j \in R} \tilde{T}_j$  {select the next component to fail}
- 37:    $T_h \leftarrow \tilde{T}_h$  {store the next failure times}
- 38:   **end for**
- 39: **end if**
- 40: **end if**
- 41: **end for**
- 42: **return**  $T_1, \dots, T_n$

**Algorithm 2:** Lifetimes simulation based on observations  $\mathbf{Y}_{obs}$



---

any two vertices):

$$\mathcal{U}_{k,d} = \{U \subseteq V; |U| = k, U \text{ is connected and } \text{diam}(U) = d\}$$

197 with  $d \leq k \leq n$  (the case with  $d = k$  is when failed components are aligned). In the sequel,  
198 we will only consider  $\mathcal{U}_k$  for defining the system TTF, but of course similar definition for  
199 the system TTF can be obtained by considering  $\mathcal{U}_{k,d}$ .

Now, let us define the system TTF  $\mathcal{T}$  of the system as follows:

$$\mathcal{T} = \min_{U \in \mathcal{U}_k} (\mathcal{T}_U).$$

200 Hence,  $\mathcal{T}$  is the first time that  $k$  connected components have all failed. We can clearly recover  
201 the classical cases of parallel and series systems. Indeed, if  $k = n$ , then  $\mathcal{U}_n = V = \{1, \dots, n\}$   
202 and thus  $\mathcal{T} = \mathcal{T}_V = \max(T_1, \dots, T_n)$ : this is a parallel system made of  $n$  identical components.  
203 If  $k = 1$ , then  $\mathcal{U}_1 = \{\{i\}, 1 \leq i \leq n\}$  is just the set of a single component and thus  $\mathcal{T}_{\{i\}} = T_i$   
204 and  $\mathcal{T} = \min(T_i)$ : this is a series system composed by  $n$  identical components.

Now the survival function of the system TTF can be derived. For any  $t \geq 0$ ,

$$\mathbb{P}[\mathcal{T} \geq t] = \mathbb{P}[\min_{U \in \mathcal{U}_k} (\mathcal{T}_U) \geq t] = \mathbb{P}[\forall U \in \mathcal{U}_k, \mathcal{T}_U \geq t].$$

205 Since  $\{\mathcal{T}_U, U \in \mathcal{U}_k\}$  is not a set of independent random variables, the survival function of  
206  $\mathcal{T}$  cannot be computed explicitly. A solution is to estimate it through simulation. However,  
207 the simulation of system TTF is also a challenging problem, since we have to detect the first  
208 time that  $k$  connected components fail. A brutal way to achieve this consists in enumerating  
209 the set of  $k$  connected components and, at each step of the simulation algorithm, to check if  
210 the failure of a new component induces that there exists such a sub-graph with only failed  
211 components. Such an enumerating task is a well known difficult problem and the number  
212 of  $k$  connected components is increasing exponentially with  $k$  (see the sequence A000088

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213 in the OEIS - On-Line Encyclopedia of Integer Sequences - for the number of graphs on  $k$   
214 unlabelled nodes). Clearly, the event of  $k$  connected failed components may arise quickly in  
215 presence of interaction between components.

216 To illustrate this approach, a numerical study will end this section. Let us consider a grid  
217 made of  $10 \times 10$  components without imposing special impact from the mechanical stress  
218 nor the other covariates (for all  $i \in V$ ,  $C_i \equiv 0$  and  $\beta = 0$ ). For the baseline distribution,  
219 parameters are set as follows:  $a = 2$  and  $b \in \{1.5, 3\}$ . For the parameter modelling the  
220 physical interaction between components, let us consider three different values:  $\alpha \in \{0, 3, 5\}$   
221 (from no interaction to strong interaction). At last,  $k \in \{4, 9, 16\}$ . However, for sake of  
222 simplicity, only square shapes are considered for the sub-graphs (instead of any connected  
223 sub-graph with  $k$  vertices). Based on 1,000 simulations, empirical means and standard  
224 deviations of system TTF are reported in Table 1. The empirical survival curves are  
225 represented in Figure 4.

226 [Table 1 about here.]

227 [Figure 4 about here.]

228 It can be noticed in Figure 4 that when parameter  $\alpha$  grows, the system lifetime decreases.  
229 Indeed,  $\alpha$  quantifies the interaction level between components in a neighbourhood. As  $\alpha$   
230 increases, the impact of a component failure on the lifetime of its neighbours becomes more  
231 significant. Therefore, for high values of  $\alpha$ , the number of failed components increases and  
232 the system failure becomes more imminent. Thus, it is logical to obtain results given in Table  
233 1 where the MTTF of the system decreases when  $\alpha$  increases.

234 Regarding parameter  $b$ , the shape parameter of the Weibull distribution (baseline hazard  
235 rate), we can notice its impact on the system MTTF in Table 1. For  $\alpha > 0$  as  $b$  grows the  
236 empirical means and standard deviations increase. In cases  $k \in \{9, 16\}$  for  $\alpha = 0$  as  $b$  grows  
237 the empirical means and standard deviations decrease.

---

## 5. NUMERICAL ILLUSTRATIONS

238 In this section, the SEM algorithm described above is illustrated on simulated data. For sake  
239 of simplicity, assume that:

- 240 • components are displayed on a  $\sqrt{n} \times \sqrt{n}$  grid;
- 241 • the inspection times are time equally spaced, say  $\tau_k = k\delta$  for some  $\delta > 0$ ,  $1 \leq k \leq m$ ;
- 242 • the baseline hazard function is defined in equation 3.

We also consider the basic model where there is no covariate, nor specific impact of the mechanical stress, but only a neighbourhood effect. In other words,  $C_{i,j} \equiv 1$  and  $\mathbf{Z}_{i,j} \equiv 0$  for any  $(i, j) \in \{1, \dots, n\}^2$ :

$$\forall t \geq 0, \quad \lambda_{i,j}(t|\mathcal{H}_{(i,j),t}) = \lambda(t) \exp \left( \alpha \sum_{(i',j') \in B_{i,j}} Y_{i',j'}(t) \right).$$

243 In Table 2, the Root mean Square Errors (RMSE) of the parameter estimates are presented  
244 for different inspection steps  $\delta \in \{0.1, 0.5\}$  and censoring rates  $\{20\%, 50\%\}$ . The real  
245 parameters are set as follows:  $a = 2$ ,  $b = 3$  and  $\alpha = 1$ . We have considered a  $20 \times 20$  grid and  
246 performed 50 iterations of the SEM algorithm. The Root mean Square Errors of estimated  
247 values have been computed by considering 50 repetitions. When the sampling is very frequent,  
248 the increase of the censoring rate has no substantial impact on the RMSE. But when  
249 the sampling is less frequent, the censoring rate can significantly degrade the RMSE. The  
250 relatively small values of RMSE bring out the performance of our estimation method.

251 [Table 2 about here.]

## 6. CONCLUDING REMARKS

252 In this paper, we have considered a model allowing a physical interaction between the  
253 components. This interaction is based on a choice of a spatial neighbourhood for the

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254 components. For such a model, we proposed a methodology to estimate the parameters using  
 255 the SEM algorithm which requires appropriate simulation algorithms. These algorithms seem  
 256 to be efficient, and they are expected to have the following behaviour: (a) the 'larger' the  
 257 neighbourhood is, the slower the algorithm reaches its stationary regime; (b) the 'larger' the  
 258 interval between two inspections is (say, as in the example studied in Section 5, the larger  
 259  $\delta$  is), the slower the algorithm converges.

260 As examples of more sophisticated models, we may consider the following ones.

- 1) Model with covariates and without covariates ( $C_i(t) = 1, i \in \mathbb{N}$ ) : consider a random time-dependent covariate as follows

$$Z_{i,j}(t) = A_{i,j}\phi(t),$$

261 where  $A_{i,j}$  are independent and identically distributed random variables and  $\phi$  is for  
 262 example one of the following deterministic functions: (a) constant:  $\phi(t) = w$ ; (b) linear:  
 263  $\phi(t) = wt$ ; (c) periodic:  $\phi(t) = w \cos\left(\frac{2\pi t}{q}\right)$ .

- 2) Model without covariates ( $\beta = 0$ ) and with more complex impact of mechanical stress: the stress may be applied at the center of the grid (for sake of simplicity, set  $n = 2p + 1$ ) and be diffused isotropically from the center as follows:

$$C_{i,j}(t) = \exp\left(-\frac{1}{\sigma^2} \left((i - p - 1)^2 + (j - p - 1)^2\right)\right).$$

- 264 3) Model with both covariates and mechanical stress: for this last case, we may combine  
 265 the covariates and the constraint introduced above.

266 This paper focused on the case of a parametric shape for the baseline hazard function.  
 267 In fact, the original Cox model is a semi-parametric model: the baseline hazard function is  
 268 a functional parameter (any positive function) while the parameters allowing to consider  
 269 covariates are Euclidean parameters. Our assumption of fully parametric model can be

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270 relaxed as it is done in the Cox model and a similar approach (as the one usually done  
271 with the Cox model) can be adopted by considering profile likelihood function (see e.g. [10]).

272 For short term perspectives, we can extend the numerical illustrations for the Time-To-  
273 Failure, etc. Other definitions of the system failure can be considered and the system TTF  
274 distribution be calculated accordingly. The impact of covariates can also be modelled through  
275 another type of models such as accelerated failure time ones.

## ACKNOWLEDGMENTS

276 This work was supported by the French National Agency of Research (ANR), project  
277 AMMSI, number ANR-2011-BS01-021.

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FIGURES

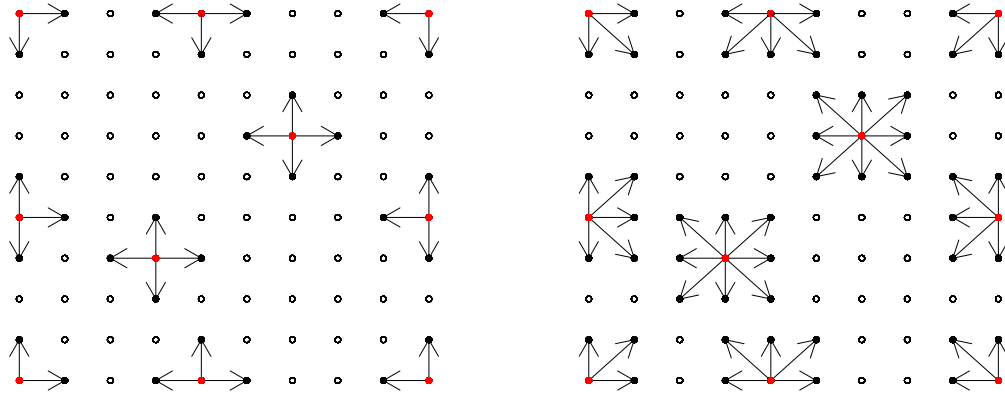
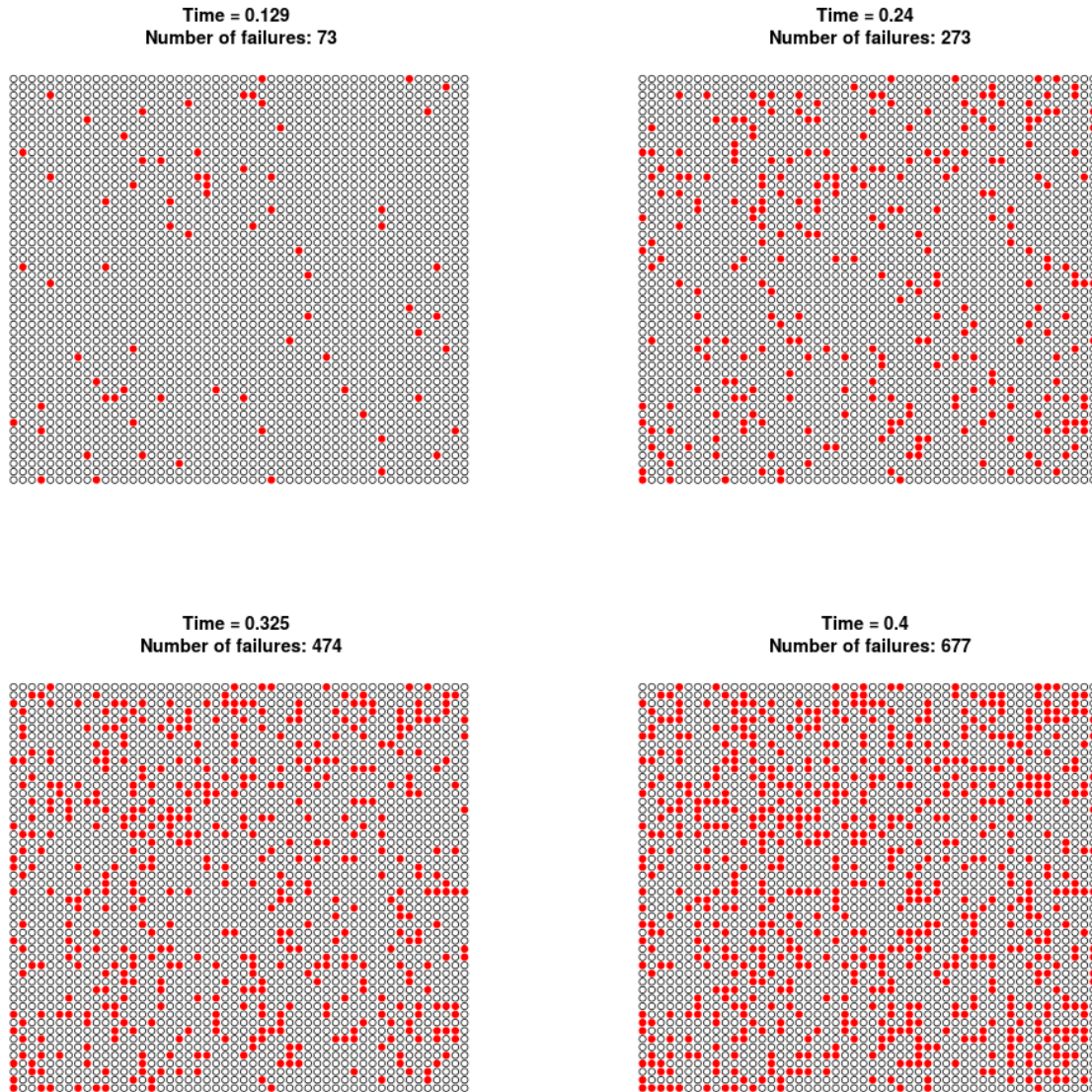


Figure 1. Illustrations of neighbourhoods: case of at most 4 neighbours (left) and case of at most 8 neighbours (right)





**Figure 2.** Case of lifetimes without physical neighbourhood interaction ( $\alpha = 0$ ): plot of failed components after 74, 274, 475 and 678 failures

FIGURES

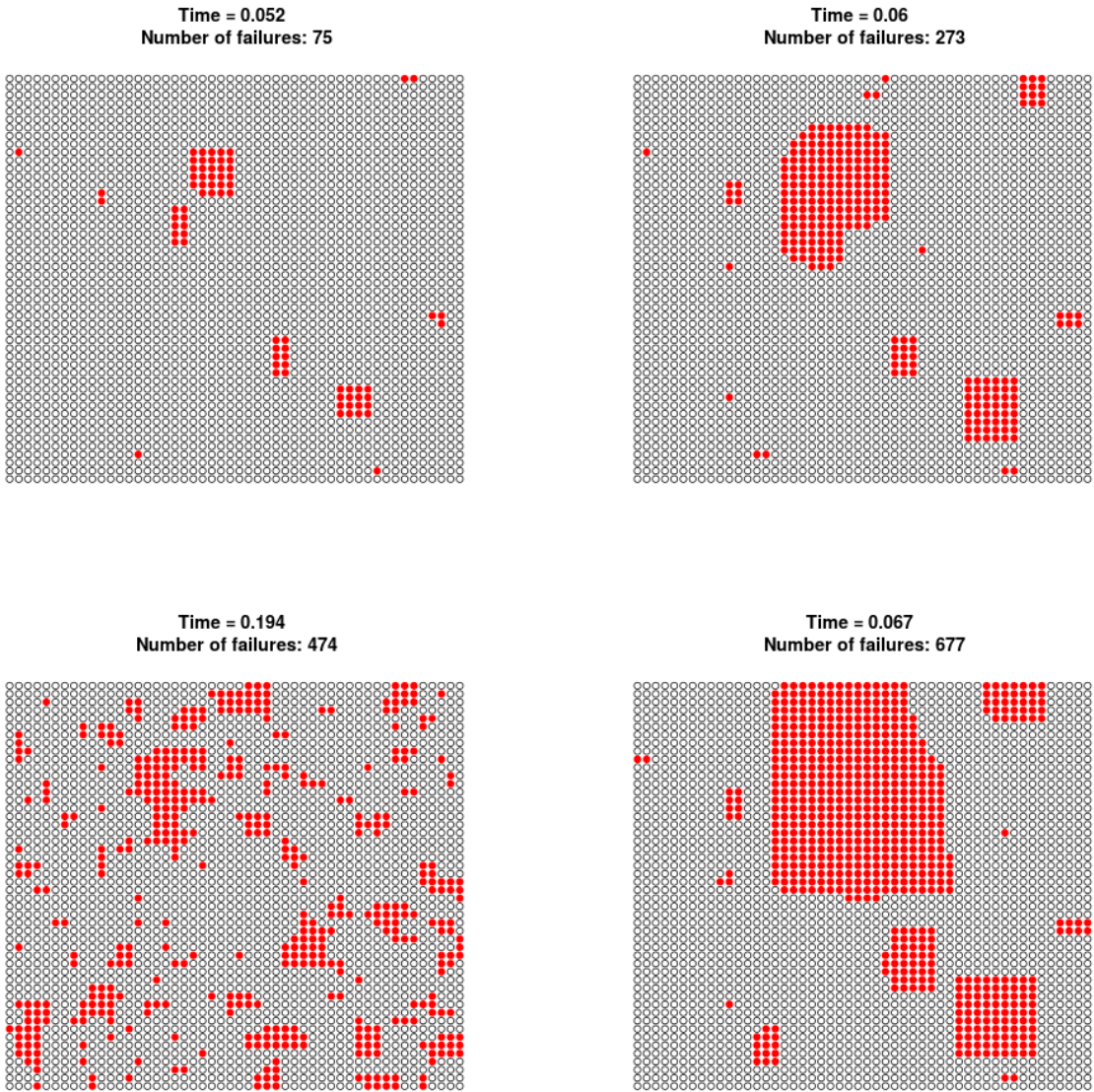


Figure 3. Case of lifetimes with physical neighbourhood interaction ( $\alpha > 0$ ): plot of failed components after 74, 274, 475 and 678 failures

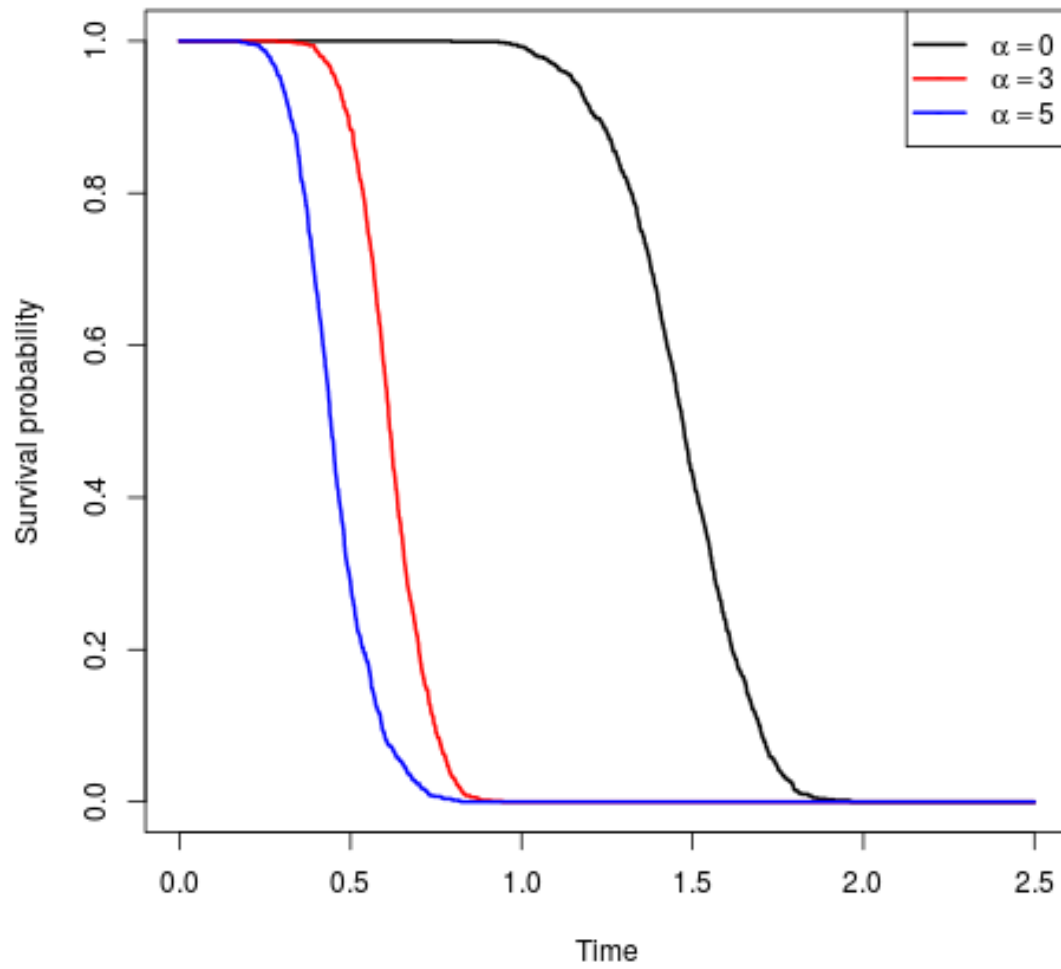


Figure 4. Empirical survival function of the system TTF with  $k = 4$ ,  $a = 2$ ,  $b = 3$  and for different values of  $\alpha$

**TABLES**

**Table 1.** Comparison of empirical means and standard deviations of system TTF for  $a = 2$  and for different values of  $\alpha$ ,  $b$  and  $k$

$k = 2 \times 2 = 4$				
	$b = 1.5$		$b = 3$	
$\alpha$	Emp. mean	Emp. std dev.	Emp. mean	Emp. std dev.
0	1.0876427	0.26016099	1.4638320	0.18031572
3	0.1945712	0.06079317	0.6161269	0.09767111
5	0.1077648	0.05126087	0.4520419	0.10582526
$k = 3 \times 3 = 9$				
	$b = 1.5$		$b = 3$	
$\alpha$	Emp. mean	Emp. std dev.	Emp. mean	Emp. std dev.
0	2.0649072	0.34523175	2.0249579	0.17143298
3	0.2382229	0.06000859	0.6847956	0.08665092
5	0.1173860	0.04992673	0.4744119	0.09856479
$k = 4 \times 4 = 16$				
	$b = 1.5$		$b = 3$	
$\alpha$	Emp. mean	Emp. std dev.	Emp. mean	Emp. std dev.
0	2.1105592	0.36572650	2.0466196	0.18027440
3	0.2560497	0.06160885	0.7104521	0.08581691
5	0.1220024	0.04954813	0.4846096	0.09574726

**Table 2.** The Root Mean Square Error of estimated values  $(\hat{a}, \hat{b}, \hat{\alpha})$  where  $a = 2$ ,  $b = 3$  and  $\alpha = 1$

censoring rate	0.2	0.5
inspection step $\delta$		
0.1	(0.3494, 0.5352, 0.1589)	(0.3562 0.5627 0.1802)
0.5	(0.9932 0.6844 0.2321)	( 1.2623 0.7665 0.2491)