



HAL
open science

A parametric lifetime model for a multicomponents system with spatial interactions

Franck Corset, Mitra Fouladirad, Christian Paroissin, Emmanuel Remy

► To cite this version:

Franck Corset, Mitra Fouladirad, Christian Paroissin, Emmanuel Remy. A parametric lifetime model for a multicomponents system with spatial interactions. *Applied Stochastic Models in Business and Industry*, 2018, 35 (2), pp.221-233. 10.1002/asmb.2449 . hal-02290382

HAL Id: hal-02290382

<https://utt.hal.science/hal-02290382v1>

Submitted on 24 May 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A parametric lifetime model for a multicomponents system with spatial interactions

F. Corset^a, M. Fouladirad^{b*}, C. Paroissin^c, E. Remy^d

Summary: In this paper, we consider a system made of n components displayed on a structure (e.g. a steel plate). We define a parametric model for the hazard function which includes covariates and spatial interaction between components. The state (non-failed or failed) of each component is observed at some inspection times. From these data, we consider the problem of model parameter estimation. To achieve this, we suggest to use the SEM algorithm based on a pseudo-likelihood function. A definition for the time-to-failure of the system is given, generalizing the classical cases. A study based on numerical simulations is provided to illustrate the proposed approach.

Keywords: Parameter estimation, Pseudo-likelihood, SEM algorithm, Spatial physical interaction

1. INTRODUCTION

1 The failure of industrial systems can cause irreversible damages, economic and human
2 losses, and also bring an unrecoverable impact on the environment. In order to avoid such
3 undesirable events, the system lifetime has to be studied, which implies to point out the
4 interactions between the components constituting the system and to propose a tractable

^a Univ. Grenoble Alpes, CNRS, Grenoble INP*, LJK, 38000 Grenoble, France.

^{*} Institute of Engineering Univ. Grenoble Alpes.

^b Institut Charles Delaunay, Université de Technologie de Troyes, CNRS UMR STMR, France.

^c CNRS / Univ Pau & Pays Adour, Laboratoire de Mathématiques et de leurs applications de Pau fédération IPRA, UMR 5142, 64000 Pau, France.

^d EDF R&D, "Performance, Industrial Risk, Monitoring for Maintenance and Operations" Department, 6 quai Watier, 78401 Chatou Cedex, France.

* Correspondence to: Institut Charles Delaunay, Université de Technologie de Troyes, CNRS UMR STMR, France; E-mail: mitra.fouladirad@utt.fr.

5 model to describe and predict the system failure. A suitable model must be complex enough
6 to take into account most of the different features of the system and simple enough to
7 be used by practitioners in the field. One of the issues in system lifetime prediction is to
8 properly take into account the system behaviour which means on the one hand the impact
9 of the environmental conditions, and on the other hand the spatial interaction between the
10 components.

11 For complex equipment, it is convenient to define the system state by a binary random
12 variable. As long as the system is operational, its health indicator is equal to one and as soon
13 as a failure occurs the health indicator is equal to zero. This model can ignore the complexity
14 of the system and makes some potentially unrealistic, but necessary, simplifications on the
15 system lifetime distribution. However, in our context, it allows to take advantage of existing
16 results on binary data analysis with covariates and spatio-temporal models. Indeed, during
17 the last decades, a great attention has been given to this subject. In Diggle et al. [9] and
18 Gumpertz et al. [14], a marginal logistic regression model is used to characterize the impact
19 of covariates on binary data. These developments are based on the notion of neighbourhood
20 which is defined as a group of components related due to their spatial interaction. After the
21 definition of what is called a neighbourhood, Besag [2, 3] proposed an auto-logistic model for
22 spatial binary data, assuming a simple dependence on surrounding neighbours which may
23 interact with each other. The interest of this model is largely pointed out in the literature for
24 instance refer to [1, 12, 16, 20, 21, 22, 25]. They all brought out its simplicity and tractability
25 in presence of data. This model is extended by Gumpertz et al. and Huffer et al. [13, 17] in
26 order to integrate covariates. Afterwards, Zhu et al. [26] give another extension of this model
27 in order to take into account the temporal dependency. Later Caragea and Kaiser [5] point out
28 that these models are not suitable in the case of a neighbourhood with a large number of failed
29 components and propose a centered model in which the failure probability of each component
30 increases or decreases around the mean behaviour of the model describing the whole system.

31 Each of these models is used in the framework of parameter inference. The natural choice
32 of estimation technique for these models is the maximum likelihood method. However, for
33 the auto-logistic model with spatially correlated variables, the hypothesis of independence of
34 the observations is no longer valid. To overcome this difficulty, Besag [4] proposes to use the
35 pseudo-likelihood which mimics the likelihood function under the hypothesis of independence
36 of the observations. Its maximization is in general easy to implement and results have good
37 statistical properties when the considered neighbourhood is not very large [24, 11]. A Monte
38 Carlo maximum likelihood method is proposed by Geyer and Thompson [11]. Alternatively,
39 Zhu et al. [26] use maximum pseudo-likelihood for parameter estimation and develop a
40 Markov Chain Monte Carlo (MCMC) algorithm for predicting future observations. Li [18]
41 studies the existence of the joint distribution for a two-step centered auto-logistic causal
42 model by considering the process as a Markov chain Markov field [15]. However, since the
43 joint distribution is untractable, he proposes the expectation-maximization pseudo-likelihood
44 estimation method and studies its performance.

45 In this paper, the model is inspired by a real industrial system installed in electric power
46 plants of EDF, the main French electric utility company. This system may be assimilated to
47 a large plate vertically maintained by screws regularly distributed all over the surface of the
48 plate. The failure of one screw leads to an increase of mechanical stress on the other screws
49 surrounding the one that has failed, consequently increasing their own failure probability.
50 When a fixed number of failed screws is reached, the system is considered to be failed in turn
51 (the plate is no more sufficiently maintained and may fall down). In addition to this spatial
52 physical interaction between the screws, environmental conditions and material properties
53 may also impact the reliability of the screws, and thus the lifetime of the whole system.

54 Since the state (failed/non-failed) of the components is known thanks to inspections, the
55 available data is binary but censored by interval and the main objective is to model both the
56 impact of the covariates and the interactive neighbourhood on the components' lifetime. In

57 reliability analysis, the impact of environmental conditions on the systems lifetime is often
58 modelled via covariates through proportional hazards or additive hazards models [7, 19, 23].
59 Therefore, in difference with the models studied in the literature and previously presented
60 in this section, we have here the constraint of using the hazard rate and we shall model the
61 impact of the spatial interaction on the failure rate of components. In this framework, we
62 integrate the censored data in the statistical inference and suggest a parameter estimation
63 procedure. Similarly to results previously exposed, a pseudo-maximum likelihood in the
64 framework of Monte Carlo simulations is proposed.

65 The remainder of this paper is organized as follows. In Section 2, the problem statement
66 and the survival model are described. Section 3 is devoted to parameter estimation and
67 statistical studies. The system failure and time-to-failure distribution are detailed in Section
68 4. In Section 5, a numerical example is considered to illustrate the proposed methodology.
69 The paper ends with a conclusion and some prospects.

2. MODEL DESCRIPTION

70 We consider a system made of n components displayed on a structure, say, for instance, a
71 set of screws on a steel plate or on a steel cylinder, see Figure 1 giving an illustration of
72 such a system. We denote by $V = \{1, \dots, n\}$ the set of all components. Let T_1, \dots, T_n be the
73 components' lifetimes. From these lifetimes, we can construct n binary stochastic processes
74 which switch from 0 to 1 after failure of the corresponding component: for any $i \in V$ and for
75 any $t \geq 0$, $Y_i(t) = \mathbf{1}_{T_i \leq t}$. The failure of a component may lead to a higher mechanical stress
76 on the components in its neighbourhood. Hence, we aim at determining if each failure of a
77 component has an impact on the neighbouring components. This impact is assumed to lead
78 to an increase of the hazard function of these components. Let B_i be the neighbourhood
79 of the i -th component. We assume that symmetry about the neighbourhood holds, in the

80 sense that if $i_1 \in B_{i_2}$ then $i_2 \in B_{i_1}$. This combination of components and neighbours can
 81 be viewed as a non-oriented graph where vertices are the components and where the edge
 82 between vertices i_1 and i_2 exists if $i_1 \in B_{i_2}$.

83 As the main example all along this paper, we assume that n is a square number and that
 84 the components are located on a regular $\sqrt{n} \times \sqrt{n}$ grid (of course, more complex topologies
 85 could be considered). For such an example, natural definitions of neighbourhood are the
 86 following ones. The first one is the case where the neighbourhood can be defined as the four
 87 cardinal points (up to the edge effect): in such a case, a component will have between 2 and 4
 88 neighbours (see Figure 1 on the left). A second example is the case where the neighbourhood
 89 can be defined as the eight cardinal points (also up to the edge effect): in such a case, a
 90 component will have between 3 and 8 neighbours (see Figure 1 on the right). Of course,
 91 other neighbourhoods could be defined and also other system structures could be considered
 92 (e.g. a cylinder). Note that sphericalness of the neighbourhood is not mandatory and, for
 93 physical reasons, such assumption can be removed.

94 [Figure 1 about here.]

The lifetime distributions of the components will be defined through their hazard functions
 by considering that covariates and spatial interactions act multiplicatively on them. Let $\mathcal{H}_{i,t}$
 be the increasing σ -field generated by all the binary stochastic processes defined above,
 except the i -th one, and up to time t :

$$\mathcal{H}_{i,t} = \sigma(Y_j(s), s \leq t, j \in \{1, \dots, n\} \setminus \{i\}).$$

95 Since $\mathcal{H}_{i,u} \subseteq \mathcal{H}_{i,t}$ for any $u \leq t$, therefore we have the following equality $\lambda_i(u|\mathcal{H}_{i,u}) =$
 96 $\lambda_i(u|\mathcal{H}_{i,t})$ for $u \leq t$.

The hazard function of the i -th component, denoted λ_i , depends on the covariates and the state of its neighbours as follows:

$$\forall t \geq 0, \quad \lambda_i(t|\mathcal{H}_{i,t}) = \lambda(t) \exp \left(\alpha \sum_{j \in B_i} C_j(t) Y_j(t) + \boldsymbol{\beta}' \mathbf{Z}_i(t) \right), \quad (1)$$

where $\lambda(\cdot)$ is the baseline hazard function, $\alpha \in \mathbb{R}$, $\boldsymbol{\beta} \in \mathbb{R}^p$ and where, for the i -th component, $\mathbf{Z}_i(\cdot)$ is a vector of p time-dependent covariates (environmental conditions, such as temperature, or material properties, such as steel plate, etc.) and $C_j(\cdot)$, $j \in B_i$, is a time-dependent covariate (it may be one component of the vector $\mathbf{Z}_j(\cdot)$, e.g. the mechanical stress). In fact, it could be sufficient to condition with respect to $\{Y_j(t), j \in B_i\}$ instead of $\mathcal{H}_{i,t}$, but it will be more convenient to condition on $\mathcal{H}_{i,t}$ when considering the cumulative hazard function and/or the conditional survival function. Indeed, the conditional survival function of the i -th component is given by:

$$\forall t \geq 0, \quad S_i(t|\mathcal{H}_{i,t}) = \exp \left(- \int_0^t \lambda_i(u|\mathcal{H}_{i,u}) du \right) \quad (2)$$

97 If $\alpha = 0$, it corresponds to the standard Cox model with time-dependent covariates.

To simplify, we assume that λ is the hazard function of the Weibull distribution with scale parameter $a > 0$ and shape parameter $b > 0$:

$$\forall t \geq 0, \quad \lambda(t) = \frac{b}{a} \left(\frac{t}{a} \right)^{b-1}. \quad (3)$$

98 However, any other parametric model could be considered for the baseline hazard function.

99 We illustrate the effect of the components' physical interaction through two simulations on
100 a square grid of 50×50 components, without imposing special impact from the mechanical
101 stress nor the other covariates (for all $i \in V$, $C_i(t) \equiv 1$ and $\boldsymbol{\beta} = 0$). The outputs are plotted
102 respectively in Figure 2 and Figure 3 (red points represent failed components). To perform
103 the simulations, we have used Algorithm 1 described in Sub-section 3.2. The parameter of

104 the Weibull hazard function was set as follows: $a = 2$ and $b = 2$. Parameter α has been set
105 to 0 (no spatial interaction) and 2 (interaction). In both cases, the figures show the state
106 of the system after 74, 274, 475 and 678 failures (for a total of 2500 components). We can
107 observe that, on contrary to the model with physically independent components, clusters of
108 failed components arise in the model with spatial interaction.

109 [Figure 2 about here.]

110 [Figure 3 about here.]

111 In the next section, we will propose an estimation procedure for the different parameters
112 involved in the model described above. The approach is based on the SEM algorithm, a
113 stochastic version of the EM algorithm proposed by Celeux et al. [6]. To this purpose, we
114 need to simulate missing observations and thus simulation algorithms will be provided.

3. STATISTICAL INFERENCE

115 Assume that the system is not often inspected and the inspections are not necessarily
116 periodical. At each control, all the components of the system are inspected. Consequently
117 the inspection times are the same for all the components: we will denote by m the number
118 of inspections and by $\tau_1 < \dots < \tau_m$ the associated deterministic times. The available data
119 at each inspection time is a binary information on whether each component is failed or
120 not. Since the failures may occur between two successive controls, the data are interval-
121 censored. Here, we neglect the possibility of non-detection of a failure during an inspection.
122 For the time-dependent covariates C_1, \dots, C_n and $\mathbf{Z}_1, \dots, \mathbf{Z}_n$, it will be assumed that they
123 are observed continuously, otherwise they will be constant piece-wise or linear piece-wise
124 interpolated.

125 In Sub-section 3.1, we provide a methodology to estimate the parameters of the model (we
 126 consider a simplified parametric model). This approach requires to simulate missing data
 127 and the useful simulation algorithms are detailed in Sub-section 3.2.

128 3.1. Parameter estimation

129 In order to proceed parameter estimation, first the expression of the likelihood function is
 130 given. It is highlighted that it is numerically too complex to use this function to estimate
 131 the model parameters. Hence, we suggest to rather use the SEM algorithm jointly with a
 132 pseudo-likelihood function. Before giving details about the proposed approach, we introduce
 133 some notations.

134 3.1.1. Notation

We shall denote by \mathbf{Y}_{obs} the set of all the observations $\{Y_i(\tau_j), i \in V, j \in \{1, \dots, m\}\}$. Notice that $\mathbf{Y}_{obs} \in \bigcup_{i=1}^m \mathcal{H}_{i, \tau_m}$. Let \mathcal{N}_1 be the set of all failed components at the last inspection time τ_m . For a component i , $1 \leq i \leq n$, let $j_i = \min\{j; Y_i(\tau_j) = 1\} \in \{1, \dots, m\}$ be the inspection time such that the failure of component i is observed for the first time. Thus, for the i -th component, the failure occurs between τ_{j_i-1} and τ_{j_i} . For convenience, we denote by I_r the set of components failing between τ_{r-1} and τ_r for any $r \in \{1, \dots, m\}$ (with the convention that $\tau_0 = 0$):

$$I_r = \{i \in \mathcal{N}_1; j_i = r\} \quad \text{and} \quad n_r = |I_r|.$$

135 Hence, we have $\mathcal{N}_1 = I_1 \cup \dots \cup I_m$. Let $I_{m+1} = V \setminus \mathcal{N}_1$ be the set of all components that
 136 have survived until the last inspection time τ_m . Let $n_{m+1} = |I_{m+1}|$. Clearly we have $n =$
 137 $n_1 + \dots + n_m + n_{m+1}$. For any $r \in \{1, \dots, m+1\}$, let $\mathcal{A}_r = \{T_i \in (\tau_{r-1}, \tau_r]; i \in I_r\}$, with
 138 $\tau_0 = 0$ and $\tau_{m+1} = +\infty$, be the set of events consisting of the components failed between
 139 τ_{r-1} and τ_r .

The likelihood function can be expressed as follows:

$$L(\alpha, \beta, a, b; \mathbf{Y}_{obs}; C_1, \dots, C_n, \mathbf{Z}_1, \dots, \mathbf{Z}_n) = \mathbb{P}[\mathcal{A}_1] \mathbb{P}[\mathcal{A}_2 | \mathcal{A}_1] \mathbb{P}[\mathcal{A}_3 | \mathcal{A}_1 \cap \mathcal{A}_2] \\ \times \dots \times \mathbb{P}[\mathcal{A}_m | \mathcal{A}_1 \cap \dots \cap \mathcal{A}_{m-1}] \mathbb{P}[\mathcal{A}_{m+1} | \mathcal{A}_1 \cap \dots \cap \mathcal{A}_m].$$

Let us compute $\mathbb{P}[\mathcal{A}_1]$:

$$\mathbb{P}[\mathcal{A}_1] = \sum_{\sigma_1 \in \Sigma_1} \mathbb{P}[0 < T_{\sigma_1(i_1)} < \dots < T_{\sigma_1(i_{n_1})} < \tau_1]$$

where Σ_1 is the set of all permutations of elements $\{i_1, \dots, i_{n_1}\}$ in I_1 . We shall recall that $|\Sigma_1| = n_1!$. Next step consists in assessing all these $n_1!$ probabilities:

$$\mathbb{P}[0 < T_{\sigma_1(i_1)} < \dots < T_{\sigma_1(i_{n_1})} < \tau_1] \\ = \int_0^{\tau_1} \mathbb{P}[0 < T_{\sigma_1(i_1)} < \dots < T_{\sigma_1(i_{n_1})} < \tau_1 | T_{\sigma_1(i_1)} = u_1] f_{T_{\sigma_1(i_1)}}(u_1) du_1 \\ = \int_0^{\tau_1} \mathbb{P}[u_1 < T_{\sigma_1(i_2)} < \dots < T_{\sigma_1(i_{n_1})} < \tau_1 | T_{\sigma_1(i_1)} = u_1] f_{T_{\sigma_1(i_1)}}(u_1) du_1$$

where

$$f_{T_{\sigma_1(i_1)}}(u_1) = \frac{b}{a} \left(\frac{u_1}{a}\right)^{b-1} \exp\left(-\left(\frac{u_1}{a}\right)^b\right).$$

Now we compute the inner probability. In the same way,

$$\mathbb{P}[u_1 < T_{\sigma_1(i_2)} < \dots < T_{\sigma_1(i_{n_1})} < \tau_1 | T_{\sigma_1(i_1)} = u_1] \\ = \int_{u_1}^{\tau_1} \mathbb{P}[u_2 < T_{\sigma_1(i_3)} < \dots < T_{\sigma_1(i_{n_1})} < \tau_1 | T_{\sigma_1(i_2)} = u_2] f_{T_{\sigma_1(i_2)} | T_{\sigma_1(i_1)}}(u_2 | u_1) du_2$$

where

$$f_{T_{\sigma_1(i_2)}|T_{\sigma_1(i_1)}}(u_2|u_1) = \lambda_{T_{\sigma_1(i_2)}|T_{\sigma_1(i_1)}}(u_2|u_1) \exp\left(-\int_0^{u_2} \lambda_{T_{\sigma_1(i_2)}|T_{\sigma_1(i_1)}}(v|u_1)dv\right)$$

with

$$\lambda_{T_{\sigma_1(i_2)}|T_{\sigma_1(i_1)}}(v|u_1) = \frac{b}{a} \left(\frac{v}{a}\right)^{b-1} \exp(\boldsymbol{\beta}' \mathbf{Z}_{i_2}(v))$$

if $i_2 \notin B_{i_1}$, and with

$$\lambda_{T_{\sigma_1(i_2)}|T_{\sigma_1(i_1)}}(v|u_1) = \begin{cases} \frac{b}{a} \left(\frac{v}{a}\right)^{b-1} \exp(\boldsymbol{\beta}' \mathbf{Z}_{i_2}(v)) & \text{for } v \leq u_1 \\ \frac{b}{a} \left(\frac{v}{a}\right)^{b-1} \exp(\alpha C_{i_1}(v) + \boldsymbol{\beta}' \mathbf{Z}_{i_2}(v)) & \text{for } v > u_1 \end{cases}$$

141 if $i_2 \in B_{i_1}$. Finally the probability $\mathbb{P}[\mathcal{A}_1]$ appears to be a n_1 -dimensional integral and thus
 142 its contribution to the likelihood function is the sum of $n_1!$ integrals in dimension n_1 . It will
 143 be similar for other contributions: for any $j \in \{1, \dots, m+1\}$, $\mathbb{P}(\mathcal{A}_j | \cap_{i=1}^{j-1} \mathcal{A}_i)$ is the sum of
 144 $n_j!$ integrals in dimension n_j .

145 Consequently two problems may arise with the direct MLE method if some of the $n_1, \dots,$
 146 n_{m+1} are too large: (1) at each step of the optimization algorithm, a numerical procedure has
 147 to be used to compute an integral in possibly high dimension; (2) a combinatorial explosion
 148 when considering the different permutations. To avoid such problems, an alternative method
 149 is proposed. Before presenting this approach, we discuss some special cases. For instance, the
 150 two previously mentioned problems will not arise if $n_j \leq 2$ for all $j \in \{1, \dots, m+1\}$. This
 151 may occur if m is close to n and if the components' lifetimes are uniformly distributed
 152 between 0 and τ_m . However, spatial physical interaction between components reinforces
 153 clusters of failed components (as shown in the simulations, see Figure 3), thus this particular
 154 case may arise essentially when there is no interaction between components.

To conclude the section, we give the likelihood function when components are physically independent (i.e., when $\alpha = 0$) and when there is no effect of the covariates (i.e., $\boldsymbol{\beta} = 0$),

since we will need it later:

$$\log L(a, b; \mathbf{Y}_{obs}) = \sum_{r=1}^m n_r \log [S(\tau_{r-1}; a, b) - S(\tau_r; a, b)] + n_{m+1} \log [S(\tau_m; a, b)] \quad (4)$$

with

$$\forall t \geq 0, \quad S(t; a, b) = \exp \left(- \left(\frac{t}{a} \right)^b \right).$$

155 3.1.3. SEM algorithm and pseudo-likelihood

156 As seen above, the maximum likelihood estimator of the parameters cannot be computed
 157 neither in a closed-form, nor numerically. Due to the censoring mechanism (failure times
 158 are never observed exactly, only interval censored data are available), we have a problem
 159 of missing observations. Indeed, hazard functions depend on binary stochastic processes
 160 Y_1, \dots, Y_n which are only observed at inspection times. A solution to this problem consists
 161 in applying the SEM algorithm. This algorithm requires to simulate the missing data, which
 162 are the times-to-failure of the components (this will be described in the next sub-section).
 163 The SEM algorithm is described below.

164 **Step 0:** parameter initialization. In equation 1, set $\hat{\alpha}^{(0)} = 0$ and $\hat{\beta}^{(0)} = 0$ (no effect).
 165 Parameters a and b can thus be estimated by maximizing the equation (4)(since
 166 components are not interacting): $\hat{a}^{(0)}$ and $\hat{b}^{(0)}$.

167 **Step k :** time-to-failures simulation and updating estimation.

- 168 1. simulate T_1, \dots, T_n using Algorithm 2 (see next Sub-section) and considering
 169 parameters $\hat{\alpha}^{(k-1)}$, $\hat{\beta}^{(k-1)}$, $\hat{a}^{(k-1)}$ and $\hat{b}^{(k-1)}$;
- 170 2. update the estimation by maximizing numerically the log-likelihood function: $\hat{\beta}^{(k)}$,
 171 $\hat{\alpha}^{(k)}$, $\hat{a}^{(k)}$ and $\hat{b}^{(k)}$.

The final estimator is then given by considering the ergodic average of the estimators. After K iterations of the algorithm, let:

$$\hat{\alpha} = \frac{1}{K} \sum_{k=1}^K \hat{\alpha}^{(k)}, \quad \hat{\beta} = \frac{1}{K} \sum_{k=1}^K \hat{\beta}^{(k)}, \quad \hat{a} = \frac{1}{K} \sum_{k=1}^K \hat{a}^{(k)} \quad \text{and} \quad \hat{b} = \frac{1}{K} \sum_{k=1}^K \hat{b}^{(k)}.$$

To finish the description of this approach, the expression of the likelihood function is provided. Algorithm 2 simulates in fact the random vector of order statistics $T_{(1)}, \dots, T_{(n)}$ (notice that the algorithm simulates also the random vector of rank statistics and let us recall that the knowledge of these two random vectors is equivalent to the knowledge of T_1, \dots, T_n). Hence, a likelihood function may be defined from such simulated data. However, a drawback of this approach is that it relies too strongly on the simulated missing observations and may fail (in fact, SEM is mainly used for unsupervised classification problem and, in such problems, only labels are not observed, so "less" data is not observed and less information is missing). For this reason, it is suggested to use the SEM algorithm based on a pseudo-likelihood function which is obtained by ignoring the dependency between observations. Pseudo-likelihood functions have been successfully considered in the literature, see, e.g., [8]. Here, we propose the following pseudo-likelihood function based on Equation (2):

$$\begin{aligned} pL(\alpha, \beta, a, b; \mathbf{Y}_{obs}, C_1, \dots, C_n, Z_1, \dots, Z_n, t_1, \dots, t_n) \\ &= \prod_{i \notin \mathcal{N}_1} S_i(\tau_m | \mathcal{H}_{i, \tau_m}) \prod_{i \in \mathcal{N}_1} [S_i(\tau_{j_i-1} | \mathcal{H}_{i, \tau_m}) - S_i(\tau_{j_i} | \mathcal{H}_{i, \tau_m})] \\ &= \prod_{i \in I_{m+1}} S_i(\tau_m | \mathcal{H}_{i, \tau_m}) \prod_{r=1}^m \prod_{i \in I_r} [S_i(\tau_{r-1} | \mathcal{H}_{i, \tau_m}) - S_i(\tau_r | \mathcal{H}_{i, \tau_m})] \end{aligned}$$

172 (a product over an empty set is considered to be equal to 1) where t_1, \dots, t_n are the missing
 173 observation which are reconstructed by using Algorithm 2. In fact, the simulated lifetimes
 174 are only used to determine the order of failure of each component.

175 **3.2. Simulation algorithms**

176 In this section, several simulation procedures are proposed. The first one will allow to simulate
 177 data that will be used in the next section for the numerical illustrations of the methodology.
 178 The second algorithm is used within the SEM algorithm.

179 *3.2.1. Data generation algorithm*

180 To generate lifetimes from the model described in Section 2, we can apply Algorithm 1. It
 181 draws the lifetimes T_1, \dots, T_n and allows to construct the stochastic processes $Y_1(\cdot), \dots, Y_n(\cdot)$.
 182 At least, to get data as considered in the previous Sub-section, we have to sample
 183 $Y_1(\cdot), \dots, Y_n(\cdot)$ at times τ_1, \dots, τ_m . To achieve this, we can stop Algorithm 1 at step r
 184 such that $T_{i_{r-1}} < \tau_m < T_{i_r}$ (it is not useful to simulate all the n lifetimes).

Require: $n, a, b, \alpha, \beta, \mathbf{Z}_1(\cdot), \dots, \mathbf{Z}_n(\cdot)$

- 1: $T \leftarrow \text{Vector}(\text{length} = n)$
- 2: $\tilde{T} \leftarrow \text{Vector}(\text{length} = n)$
- 3: draw U_1, \dots, U_n i.i.d. from the uniform distribution over $[0; 1]$
- 4: **{Step 1}**
- 5: set $R \leftarrow \{1, \dots, n\}$ **{set of non-failed components}**
- 6: draw $\tilde{T}_1, \dots, \tilde{T}_n$ by solving the following equation with respect to t_1, \dots, t_n :

$$\log(U_i) + \int_0^{t_i} \lambda(s) \exp(\beta' \mathbf{Z}_i(s)) ds = 0$$

- 7: $i_1 \leftarrow \text{argmin}_{j \in R} \tilde{T}_j$ **{select the next component to fail}**
- 8: $T_{i_1} \leftarrow \tilde{T}_{i_1}$ **{store the next failure times}**
- 9: **for** $r \in \{2, \dots, n\}$ **do**
- 10: **{Step r }**
- 11: $R \leftarrow R \setminus \{i_{r-1}\}$ **{update the set of non-failed components}**
- 12: **for** $j \in B_{i_{r-1}} \cap R$ **do**
- 13: draw \tilde{T}_j by solving the following equation with respect to t_j :

$$\log(U_j) + \int_{T_{i_{r-1}}}^{t_j} \lambda(s) \exp \left(\alpha \sum_{k \in B_j} C_k(s) Y_k(s) + \beta' \mathbf{Z}_j(s) \right) ds = 0$$

- 14: **end for**
- 15: $i_r \leftarrow \text{argmin}_{j \in R} \tilde{T}_j$ **{select the next component to fail}**
- 16: $T_{i_r} \leftarrow \tilde{T}_{i_r}$ **{store the next failure times}**
- 17: **end for**
- 18: **return** T_1, \dots, T_n

Algorithm 1: Model simulation

185 *3.2.2. Simulation algorithm using observation data*

186 In this paragraph, the general idea of the algorithm required to use the SEM algorithm
187 is described. Roughly speaking, the idea is to simulate data by considering the successive
188 observation intervals. Recall that, for any $j \in \{1, \dots, m\}$, I_j is the set of components whose
189 lifetimes are between τ_{j-1} and τ_j and I_{m+1} is the set of non-failed components at the last
190 inspection time τ_m . The procedure starts by simulating lifetimes for components in I_1 , then
191 in I_2 , and so on. Algorithm 2 describes the suggested approach.

4. SYSTEM TIME-TO-FAILURE

192 A definition of the system time-to-failure (system TTF, in short) can be based on a spatial
193 extension of consecutive k out of n systems. In such a model, when components are displayed
194 on a line (which appears to be a special case of our model) or on a circle, the system
195 is assumed to be failed as soon as k consecutive components have failed. To achieve this
196 generalization, several notations and random variables are introduced.

Let us recall that $V = \{1, \dots, n\}$ denotes the set of all components. For any subset $U \subseteq V$,
set $\mathcal{T}_U = \max_{i \in U}(T_i)$. It corresponds to the time such that all components in U have failed.
Next, for any $k \in \{1, \dots, n\}$, let :

$$\mathcal{U}_k = \{U \subseteq V; |U| = k \text{ and } U \text{ is connected}\}$$

be the set of all subsets made of k connected components. We recall that two components
are connected if they are neighbours (in other words, if there exists an edge between these
two components) and that a sub-graph is connected if there exists a path between any two
components in this sub-graph. Imposing the connectivity of U is important due to the spatial
interaction of the components. We could also impose a condition on the diameter of the sub-
graph as follows (recall that the diameter of a graph is the longest shortest path between

Require: $n, a, b, \alpha, \beta, \mathbf{Z}_1(\cdot), \dots, \mathbf{Z}_n(\cdot), \mathbf{Y}_{obs}$

- 1: $T \leftarrow \text{Vector}(\text{length} = n)$
- 2: $\tilde{T} \leftarrow \text{Vector}(\text{length} = n)$
- 3: draw U_1, \dots, U_n i.i.d. from the uniform distribution over $[0; 1]$
- 4: $\{\text{Interval } I_1\}$
- 5: $R \leftarrow I_1$ {set of non-failed components in I_1 }
- 6: **for** $j \in I_1$ **do**
- 7: draw \tilde{T}_j by solving the following equation with respect to t_j :

$$\log(U_j) + \int_0^{t_j} \lambda(s) \exp(\beta' \mathbf{Z}_j(s)) ds = 0$$

- 8: **end for**
- 9: $h \leftarrow \text{argmin}_{j \in R} \tilde{T}_j$ {select the next component to fail}
- 10: $T_h \leftarrow \tilde{T}_h$ {store the next failure times}
- 11: **if** $n_1 > 1$ **then**
- 12: **for** $r \in \{2, \dots, n_1\}$ **do**
- 13: $R \leftarrow R \setminus \{h\}$ {update the set of non-failed components in I_1 }
- 14: **for** $j \in B_h \cap R$ **do**
- 15: draw \tilde{T}_j by solving the following equation with respect to t_j :

$$\log(U_j) + \int_{T_h}^{t_j} \lambda(s) \exp \left(\alpha \sum_{k \in B_j} C_k(s) Y_k(s) + \beta' \mathbf{Z}_j(s) \right) ds = 0$$

- 16: **end for**
- 17: $h \leftarrow \text{argmin}_{j \in R} \tilde{T}_j$ {select the next component to fail}
- 18: $T_h \leftarrow \tilde{T}_h$ {store the next failure times}
- 19: **end for**
- 20: **end if**
- 21: **for** $i \in \{2, \dots, m\}$ **do**
- 22: {Interval I_i }
- 23: $R \leftarrow I_i$
- 24: **if** $n_i > 0$ **then**
- 25: **for** $j \in I_i$ **do**
- 26: draw \tilde{T}_j by solving the following equation with respect to t_j :

$$\log(U_j) + \int_{\tau_{i-1}}^{t_j} \lambda(s) \exp \left(\alpha \sum_{k \in B_j} C_k(s) Y_k(s) + \beta' \mathbf{Z}_j(s) \right) ds = 0$$

- 27: **end for**
- 28: $h \leftarrow \text{argmin}_{j \in R} \tilde{T}_j$ {select the next component to fail}
- 29: $T_h \leftarrow \tilde{T}_h$ {store the next failure times}
- 30: **if** $n_i > 1$ **then**
- 31: **for** $r \in \{2, \dots, n_i\}$ **do**
- 32: $R \leftarrow R \setminus \{h\}$ {update the set of non-failed components in I_i }
- 33: **for** $j \in B_h \cap R$ **do**
- 34: draw \tilde{T}_j by solving the following equation with respect to t_j :

$$\log(U_j) + \int_{T_h}^{t_j} \lambda(s) \exp \left(\alpha \sum_{k \in B_j} C_k(s) Y_k(s) + \beta' \mathbf{Z}_j(s) \right) ds = 0$$

- 35: **end for**
- 36: $h \leftarrow \text{argmin}_{j \in R} \tilde{T}_j$ {select the next component to fail}
- 37: $T_h \leftarrow \tilde{T}_h$ {store the next failure times}
- 38: **end for**
- 39: **end if**
- 40: **end if**
- 41: **end for**
- 42: **return** T_1, \dots, T_n

Algorithm 2: Lifetimes simulation based on observations \mathbf{Y}_{obs}

any two vertices):

$$\mathcal{U}_{k,d} = \{U \subseteq V; |U| = k, U \text{ is connected and } \text{diam}(U) = d\}$$

197 with $d \leq k \leq n$ (the case with $d = k$ is when failed components are aligned). In the sequel,
198 we will only consider \mathcal{U}_k for defining the system TTF, but of course similar definition for
199 the system TTF can be obtained by considering $\mathcal{U}_{k,d}$.

Now, let us define the system TTF \mathcal{T} of the system as follows:

$$\mathcal{T} = \min_{U \in \mathcal{U}_k} (\mathcal{T}_U).$$

200 Hence, \mathcal{T} is the first time that k connected components have all failed. We can clearly recover
201 the classical cases of parallel and series systems. Indeed, if $k = n$, then $\mathcal{U}_n = V = \{1, \dots, n\}$
202 and thus $\mathcal{T} = \mathcal{T}_V = \max(T_1, \dots, T_n)$: this is a parallel system made of n identical components.
203 If $k = 1$, then $\mathcal{U}_1 = \{\{i\}, 1 \leq i \leq n\}$ is just the set of a single component and thus $\mathcal{T}_{\{i\}} = T_i$
204 and $\mathcal{T} = \min(T_i)$: this is a series system composed by n identical components.

Now the survival function of the system TTF can be derived. For any $t \geq 0$,

$$\mathbb{P}[\mathcal{T} \geq t] = \mathbb{P}[\min_{U \in \mathcal{U}_k} (\mathcal{T}_U) \geq t] = \mathbb{P}[\forall U \in \mathcal{U}_k, \mathcal{T}_U \geq t].$$

205 Since $\{\mathcal{T}_U, U \in \mathcal{U}_k\}$ is not a set of independent random variables, the survival function of
206 \mathcal{T} cannot be computed explicitly. A solution is to estimate it through simulation. However,
207 the simulation of system TTF is also a challenging problem, since we have to detect the first
208 time that k connected components fail. A brutal way to achieve this consists in enumerating
209 the set of k connected components and, at each step of the simulation algorithm, to check if
210 the failure of a new component induces that there exists such a sub-graph with only failed
211 components. Such an enumerating task is a well known difficult problem and the number
212 of k connected components is increasing exponentially with k (see the sequence A000088

213 in the OEIS - On-Line Encyclopedia of Integer Sequences - for the number of graphs on k
214 unlabelled nodes). Clearly, the event of k connected failed components may arise quickly in
215 presence of interaction between components.

216 To illustrate this approach, a numerical study will end this section. Let us consider a grid
217 made of 10×10 components without imposing special impact from the mechanical stress
218 nor the other covariates (for all $i \in V$, $C_i \equiv 0$ and $\beta = 0$). For the baseline distribution,
219 parameters are set as follows: $a = 2$ and $b \in \{1.5, 3\}$. For the parameter modelling the
220 physical interaction between components, let us consider three different values: $\alpha \in \{0, 3, 5\}$
221 (from no interaction to strong interaction). At last, $k \in \{4, 9, 16\}$. However, for sake of
222 simplicity, only square shapes are considered for the sub-graphs (instead of any connected
223 sub-graph with k vertices). Based on 1,000 simulations, empirical means and standard
224 deviations of system TTF are reported in Table 1. The empirical survival curves are
225 represented in Figure 4.

226 [Table 1 about here.]

227 [Figure 4 about here.]

228 It can be noticed in Figure 4 that when parameter α grows, the system lifetime decreases.
229 Indeed, α quantifies the interaction level between components in a neighbourhood. As α
230 increases, the impact of a component failure on the lifetime of its neighbours becomes more
231 significant. Therefore, for high values of α , the number of failed components increases and
232 the system failure becomes more imminent. Thus, it is logical to obtain results given in Table
233 1 where the MTTF of the system decreases when α increases.

234 Regarding parameter b , the shape parameter of the Weibull distribution (baseline hazard
235 rate), we can notice its impact on the system MTTF in Table 1. For $\alpha > 0$ as b grows the
236 empirical means and standard deviations increase. In cases $k \in \{9, 16\}$ for $\alpha = 0$ as b grows
237 the empirical means and standard deviations decrease.

5. NUMERICAL ILLUSTRATIONS

238 In this section, the SEM algorithm described above is illustrated on simulated data. For sake
239 of simplicity, assume that:

- 240 • components are displayed on a $\sqrt{n} \times \sqrt{n}$ grid;
- 241 • the inspection times are time equally spaced, say $\tau_k = k\delta$ for some $\delta > 0$, $1 \leq k \leq m$;
- 242 • the baseline hazard function is defined in equation 3.

We also consider the basic model where there is no covariate, nor specific impact of the mechanical stress, but only a neighbourhood effect. In other words, $C_{i,j} \equiv 1$ and $\mathbf{Z}_{i,j} \equiv 0$ for any $(i, j) \in \{1, \dots, n\}^2$:

$$\forall t \geq 0, \quad \lambda_{i,j}(t|\mathcal{H}_{(i,j),t}) = \lambda(t) \exp \left(\alpha \sum_{(i',j') \in B_{i,j}} Y_{i',j'}(t) \right).$$

243 In Table 2, the Root mean Square Errors (RMSE) of the parameter estimates are presented
244 for different inspection steps $\delta \in \{0.1, 0.5\}$ and censoring rates $\{20\%, 50\%\}$. The real
245 parameters are set as follows: $a = 2$, $b = 3$ and $\alpha = 1$. We have considered a 20×20 grid and
246 performed 50 iterations of the SEM algorithm. The Root mean Square Errors of estimated
247 values have been computed by considering 50 repetitions. When the sampling is very frequent,
248 the increase of the censoring rate has no substantial impact on the RMSE. But when
249 the sampling is less frequent, the censoring rate can significantly degrade the RMSE. The
250 relatively small values of RMSE bring out the performance of our estimation method.

251 [Table 2 about here.]

6. CONCLUDING REMARKS

252 In this paper, we have considered a model allowing a physical interaction between the
253 components. This interaction is based on a choice of a spatial neighbourhood for the

254 components. For such a model, we proposed a methodology to estimate the parameters using
 255 the SEM algorithm which requires appropriate simulation algorithms. These algorithms seem
 256 to be efficient, and they are expected to have the following behaviour: (a) the 'larger' the
 257 neighbourhood is, the slower the algorithm reaches its stationary regime; (b) the 'larger' the
 258 interval between two inspections is (say, as in the example studied in Section 5, the larger
 259 δ is), the slower the algorithm converges.

260 As examples of more sophisticated models, we may consider the following ones.

- 1) Model with covariates and without covariates ($C_i(t) = 1, i \in \mathbb{N}$) : consider a random time-dependent covariate as follows

$$Z_{i,j}(t) = A_{i,j}\phi(t),$$

261 where $A_{i,j}$ are independent and identically distributed random variables and ϕ is for
 262 example one of the following deterministic functions: (a) constant: $\phi(t) = w$; (b) linear:
 263 $\phi(t) = wt$; (c) periodic: $\phi(t) = w \cos\left(\frac{2\pi t}{q}\right)$.

- 2) Model without covariates ($\beta = 0$) and with more complex impact of mechanical stress: the stress may be applied at the center of the grid (for sake of simplicity, set $n = 2p + 1$) and be diffused isotropically from the center as follows:

$$C_{i,j}(t) = \exp\left(-\frac{1}{\sigma^2}((i-p-1)^2 + (j-p-1)^2)\right).$$

- 264 3) Model with both covariates and mechanical stress: for this last case, we may combine
 265 the covariates and the constraint introduced above.

266 This paper focused on the case of a parametric shape for the baseline hazard function.
 267 In fact, the original Cox model is a semi-parametric model: the baseline hazard function is
 268 a functional parameter (any positive function) while the parameters allowing to consider
 269 covariates are Euclidean parameters. Our assumption of fully parametric model can be

270 relaxed as it is done in the Cox model and a similar approach (as the one usually done
271 with the Cox model) can be adopted by considering profile likelihood function (see e.g. [10]).

272 For short term perspectives, we can extend the numerical illustrations for the Time-To-
273 Failure, etc. Other definitions of the system failure can be considered and the system TTF
274 distribution be calculated accordingly. The impact of covariates can also be modelled through
275 another type of models such as accelerated failure time ones.

ACKNOWLEDGMENTS

276 This work was supported by the French National Agency of Research (ANR), project
277 AMMSI, number ANR-2011-BS01-021.

REFERENCES

- 278 [1] M. S. Bartlett. The statistical analysis of spatial pattern. Advances in Applied Probability Cambridge
279 University Press, 6(2):336–358, 1974.
- 280 [2] J. Besag. Nearest-neighbour systems and the auto-logistic model for binary data. Journal of the Royal
281 Statistical Society. Series B (Methodological), 34:75–83, 1972.
- 282 [3] J. Besag. Spatial interaction and the statistical analysis of lattice systems. Journal of the Royal
283 Statistical Society. Series B (Methodological), pages 192–236, 1974.
- 284 [4] J. Besag. Statistical analysis of non-lattice data. The Statistician, 24:179–195, 1975.
- 285 [5] P. C. Caragea and M. S. Kaiser. Autologistic models with interpretable parameters. Journal of
286 agricultural, biological, and environmental statistics, 14(3):281–300, 2009.
- 287 [6] G. Celeux, D. Chauveau, and J. Diebolt. Stochastic versions of the EM algorithm: an experimental
288 study in the mixture case. Journal of Statistical Computation and Simulation, 4(55):287–314, 1996.
- 289 [7] D.R. Cox. Regression models and life-tables. Journal of the Royal Statistical Society. Series B
290 (Methodological), 34(2):187–220, 1972.
- 291 [8] D.R. Cox and N. Reid. Miscellanea. A note on pseudolikelihood constructed from marginal densities.
292 Biometrika, 3(91):729–737, 2004.

-
- 293 [9] P. Diggle, K.Y. Liang, and S.L. Zeger. Analysis of longitudinal data. Clarendon Press, Oxford University
294 Press, Oxford, New York, 1994.
- 295 [10] Cox D.R. Principles of statistical inference. Cambridge University Press, Cambridge, 2006.
- 296 [11] C. J. Geyer and E. A. Thompson. Constrained Monte Carlo maximum likelihood for dependent data.
297 Journal of the Royal Statistical Society. Series B (Methodological), 54(3):657–699, 1991.
- 298 [12] Daniel A Griffith. A spatial filtering specification for the autologistic model. Environment and Planning
299 A: Economy and Space, 36(10):1791–1811, 2004/01/02 2004.
- 300 [13] M. L. Gumpertz, J. M. Graham, and J. B. Ristaino. Autologistic model of spatial pattern of
301 phytophthora epidemic in bell pepper: effects of soil variables on disease presence. Journal of
302 Agricultural, Biological, and Environmental Statistics, pages 131–156, 1997.
- 303 [14] M. L. Gumpertz, C. T. Wu, and J. M. Pye. Logistic regression for southern pine beetle outbreaks with
304 spatial and temporal autocorrelation. Forest Science, 46:95–107, 2000.
- 305 [15] X. Guyon and C. Hardouin. Markov chain Markov field dynamics: models and statistics. Statistics: A
306 Journal of Theoretical and Applied Statistics, 36(4):339–363, 2002.
- 307 [16] J. A. Hoeting, M. Leecaster, and D. Bowden. An improved model for spatially correlated binary
308 responses. International Biometric Society, Springer, 5(1):102–114, 2000.
- 309 [17] F. W. Huffer and H. Wu. Markov chain monte carlo for autologistic regression models with application
310 to the distribution of plant species. Biometrics, pages 509–524, 1998.
- 311 [18] S. Li. Modélisation spatio-temporelle pour l’esca de la vigne à l’échelle de la parcelle. PhD thesis,
312 Université de Bordeaux, 2015.
- 313 [19] M. J. Newby. Perspective on Weibull proportional-hazards model. IEEE Transactions on Reliability,
314 43(2):217–223, 1994.
- 315 [20] Keith Ord. Estimation methods for models of spatial interaction. Journal of the American Statistical
316 Association, 70(349):120–126, 1975.
- 317 [21] J. F. Pendergast, S. J. Gange, M. A. Newton, M. J. Lindstrom, M. Palta, and M. R. Fisher. A survey
318 of methods for analyzing clustered binary response data. International Statistical Review / Revue
319 Internationale de Statistique, 64(1):89–118, 1996.
- 320 [22] Michael Sherman, Tatiyana V. Apanasovich, and Raymond J. Carroll. On estimation in binary
321 autologistic spatial models. Journal of Statistical Computation and Simulation, 76(2):167–179, 2006.
- 322 [23] N. D. Singpurwalla. Survival in dynamic environments. Statistical Science, 1(10):86–103, 1995.

-
- 323 [24] D. Strauss. The many faces of logistic regression. The American Statistician, 46(4):321–327, 1992.
- 324 [25] H. Wu and F.R.W. Huffer. Modelling the distribution of plant species using the autologistic regression
325 model. Environmental and Ecological Statistics, 4(1):31–48, 1997.
- 326 [26] J. Zhu, H.-C. Huang, and J. Wu. Modeling spatial-temporal binary data using Markov random fields.
327 Journal of Agricultural, Biological, and Environmental Statistics, 10(2):212–225, 2005.

FIGURES

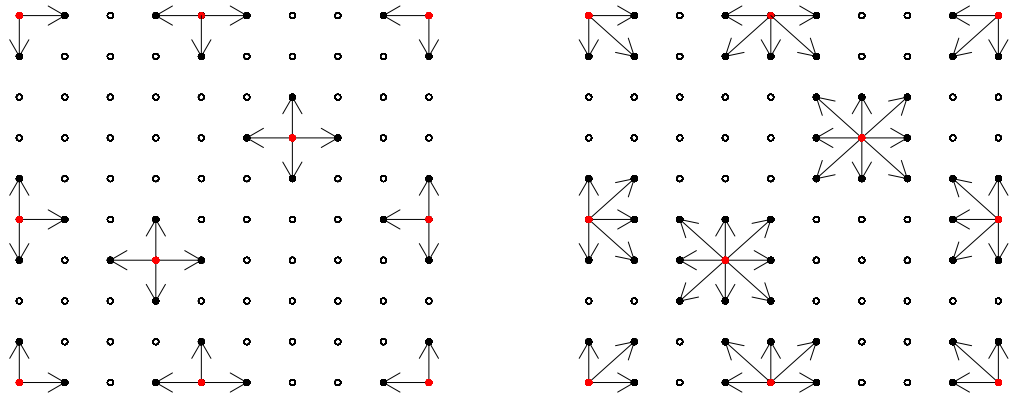


Figure 1. Illustrations of neighbourhoods: case of at most 4 neighbours (left) and case of at most 8 neighbours (right)

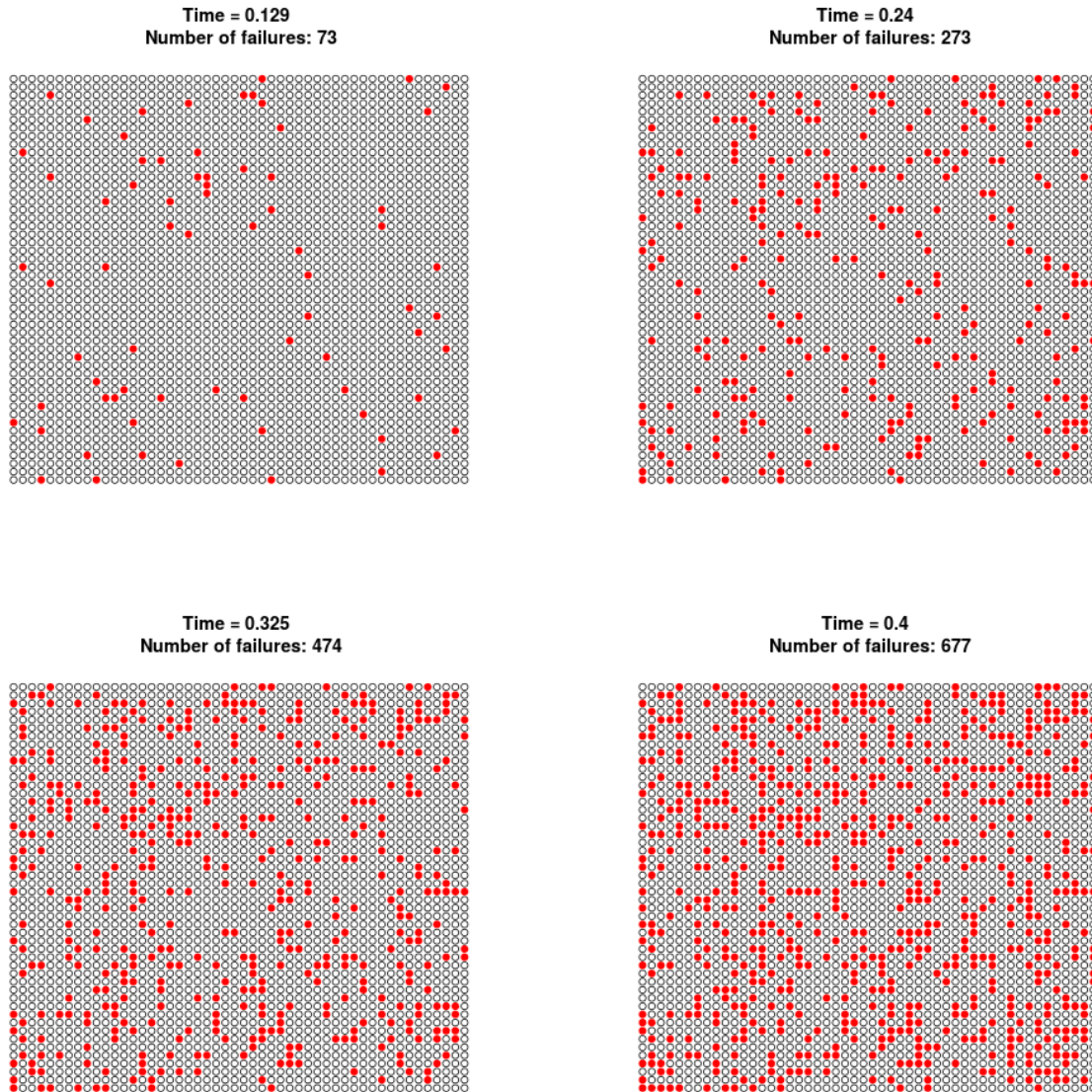


Figure 2. Case of lifetimes without physical neighbourhood interaction ($\alpha = 0$): plot of failed components after 74, 274, 475 and 678 failures

FIGURES

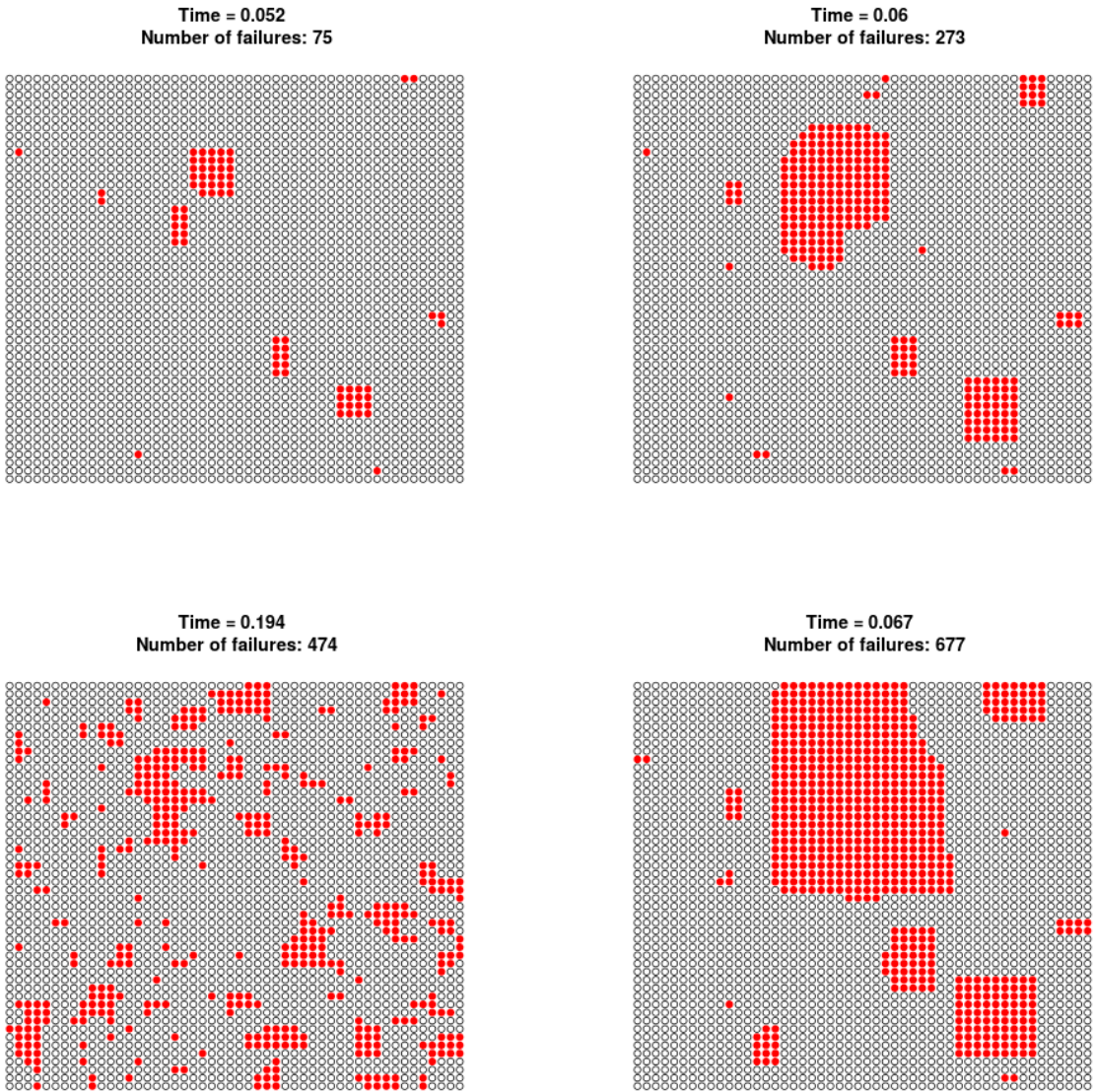


Figure 3. Case of lifetimes with physical neighbourhood interaction ($\alpha > 0$): plot of failed components after 74, 274, 475 and 678 failures

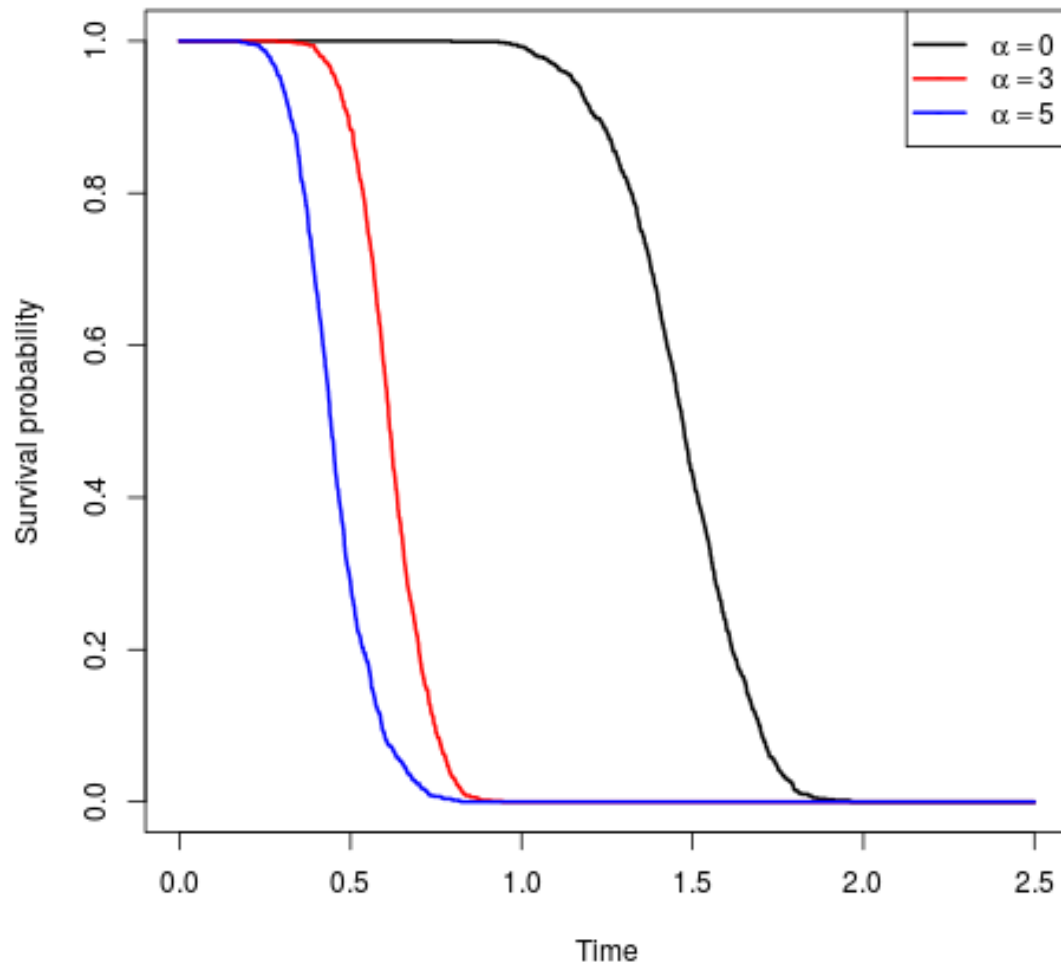


Figure 4. Empirical survival function of the system TTF with $k = 4$, $a = 2$, $b = 3$ and for different values of α

TABLES

Table 1. Comparison of empirical means and standard deviations of system TTF for $a = 2$ and for different values of α , b and k

$k = 2 \times 2 = 4$				
	$b = 1.5$		$b = 3$	
α	Emp. mean	Emp. std dev.	Emp. mean	Emp. std dev.
0	1.0876427	0.26016099	1.4638320	0.18031572
3	0.1945712	0.06079317	0.6161269	0.09767111
5	0.1077648	0.05126087	0.4520419	0.10582526
$k = 3 \times 3 = 9$				
	$b = 1.5$		$b = 3$	
α	Emp. mean	Emp. std dev.	Emp. mean	Emp. std dev.
0	2.0649072	0.34523175	2.0249579	0.17143298
3	0.2382229	0.06000859	0.6847956	0.08665092
5	0.1173860	0.04992673	0.4744119	0.09856479
$k = 4 \times 4 = 16$				
	$b = 1.5$		$b = 3$	
α	Emp. mean	Emp. std dev.	Emp. mean	Emp. std dev.
0	2.1105592	0.36572650	2.0466196	0.18027440
3	0.2560497	0.06160885	0.7104521	0.08581691
5	0.1220024	0.04954813	0.4846096	0.09574726

Table 2. The Root Mean Square Error of estimated values $(\hat{a}, \hat{b}, \hat{\alpha})$ where $a = 2$, $b = 3$ and $\alpha = 1$

censoring rate	0.2	0.5
inspection step δ		
0.1	(0.3494, 0.5352, 0.1589)	(0.3562 0.5627 0.1802)
0.5	(0.9932 0.6844 0.2321)	(1.2623 0.7665 0.2491)