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Abstract

With the development of monitoring equipment, research on condition-based maintenance (CBM) is rapidly growing. CBM optimization aims to find an optimal CBM policy which minimizes the average cost of the system over a specified duration of time. This paper proposes a dynamic auto-adaptive predictive maintenance policy for single-unit systems whose gradual deterioration is governed by an increasing stochastic process. The parameters of the degradation process are assumed to be unknown and Bayes' theorem is used to update the prior information. The time interval between two successive inspections is scheduled based on the remaining useful life (RUL) of the system and is updated along with the degradation parameters. A procedure is proposed to dynamically adapt the maintenance decision variables accordingly. Finally, different possible maintenance policies are considered and compared to illustrate their performance.

Keywords: Maintenance, Condition-based, Remaining useful life, Increasing stochastic process, Bayesian update.

1. Introduction

Maintenance decision making is of prime importance for improving the global performance of industrial systems and structures during their useful life. To date, many maintenance and replacement models, from simple age-based to more sophisticated condition-based, have been proposed. Wang [24] made a clear classification of different possible strategies. The tendency towards condition-

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based maintenance (CBM) policies is mainly due to a great improvement in technology which enables accurate online measurements of degradation levels and leads to benefits over traditional time-based maintenance [12].

A recent review of existing maintenance policies can be found in the paper of Alaswad and Xiang [1] which also classifies CBM policies depending on the deterioration model. The major part of their paper discussed maintenance strategies in which the degradation evolution is described by stochastic processes; the Wiener, the gamma and the inverse Gaussian (IG) processes received special mention. Elwany et al. [6], Guo et al. [11], and Zhang et al. [27] investigated examples of CBM policies based on the Wiener process. A vast amount of literature is devoted to the use of the gamma process in maintenance modeling; see, e.g., Grall et al. [10], Dieulle et al. [4], Tan et al. [22], Mercier et al. [19], and van Noortwijk [23]. The IG process has been considered more recently in degradation modeling. Chen et al. [3] investigated the optimal CBM policy with periodic inspections when the system degradation follows an IG process with a random-drift model, while Li et al. [17] obtained the optimal CBM strategy by maximizing the availability under cost constraints.

A common, but usually false assumption in maintenance literature is that the parameters of the model are known. In fact the model parameters are usually unknown and must be estimated, see, e.g., [3, 15, 27]. De Jonge et al. [13] considered time-based preventive maintenance under uncertainty in lifetime distribution and analyzed the effect of postponing maintenance actions on cost. The case when the degradation parameters change after a change in the system degradation rate and the new parameters are unknown was considered by Fouladirad and Grall [8]. Kallen and van Noortwijk [15] employed the Bayes method to estimate the unknown degradation parameters. Articles on iterative Bayesian updates of degradation parameters can be found in Gebraeel et al. [9], Xu and Wang [25], and Bian and Gebraeel [2]. Examples of adaptive preventive maintenance policies using Weibull distribution with the help of Bayes' theorem can be found in [14, 21]. In the framework of dynamic condition-based maintenance, Flage et al. [7] gave a general description of the case of unknown degradation parameters with Bayesian updates and optimization of decision variables after inspections. The authors introduced and emphasize on safety constraints in decision variable optimization throughout the paper. In their study, they employed a discrete stochastic process for modeling degradation which leads to i-fold integrals for calculating the maintenance cost.

In the same framework as in [7] and to avoid overwhelming calculations, we propose a heuristic way to assess the maintenance cost. The present work considers a system subject to degradation when the degradation evolution follows a non-decreasing continuous-state stochastic process. The main contribution of the present work is to propose an aperiodic condition-based maintenance policy when the parameters of the model are unknown. All the information about these parameters (expert judgments or previous tests) may be modeled by a prior distribution. The RUL function allows us to assume such aperiodic inspection planning beside to having a high reliability (safety) level. The special case of IG process is developed because of analytical tractability in a Bayesian framework. We show how to employ Bayes' theorem to obtain successive updates for the posterior distribution of the parameters, thus increasing knowledge of the model parameters with inspection results. We also study the effect of these updates on the RUL function, used in inspection planning, as well as on the behavior of the cost function.

The remainder of the paper is organized as follows. Section 2 provides a basic description of the system and states the main assumptions of the maintenance policy. A static aperiodic maintenance policy is studied in Section 3 for the cases of both known and unknown model parameters. Section 4 describes the way the procedure of Bayesian updating is considered in adapting current information and making the maintenance decision rule dynamic and auto-adaptive in case of unknown degradation parameters. Section 5 gives more insight related to an implementation with the IG degradation process. In order to demonstrate the validity of the dynamic auto-adaptive predictive maintenance policy, different maintenance policies are compared in Section 6. Finally, Section 7 contains a conclusion.

2. System and Maintenance Decision Rule: Descriptions and Main Assumptions

Consider a single component system subject to wear or deterioration. Let X_t denote the degradation state of the system at time t. In the absence of repair or replacement actions, the evolution of the system deterioration is assumed to be strictly increasing. This property and the time dependency structure of X_t is a major motivation for employing a monotone increasing stochastic process to model the degradation evolution. We shall assume that the initial degradation state of the system is zero; i.e. $X_0 = 0$.

It is assumed that continuous monitoring is not possible for technological or economic reasons; the system condition can only be observed by inspection. Moreover, it is also assumed that inspec-

tions are instantaneous, perfect and non-destructive. The system fails when its degradation level reaches or crosses a pre-specified threshold level L, chosen according to the system characteristics. A failure is not self-announcing and can only be detected by inspection.

2.1. Stochastic Degradation Process

We assume that the stochastic degradation process X_t can be modeled by a monotone increasing stochastic process such that

1. $X_0 = 0$ with probability one;

2. Non-overlapping increments of X_t are statistically independent;

3. The process X_t has stationary increments i.e. the distribution of $X(t+s) - X(s)$ depends only on the length t of the interval but not on s. We refer to the pdf (resp. cdf) of the distribution of X_t by $f_{\Theta(t)}(x)$ (resp. $F_{\Theta(t)}(x)$). The function Θ is the vector of distribution parameters. For the sake of simplicity, when t is omitted Θ will denote the vector of parameters which characterizes the function $\Theta(t)$.

As mentioned above, once degradation exceeds the threshold L, failure occurs and corrective action is necessary. Hence the lifetime of a product is simply the time of first passage of X_t across the threshold L. At a given time t, this lifetime is fully determined by the distribution of the time until the degradation signal reaches the failure threshold, given the knowledge of the condition of the system at time t. Hence, given the parameters, we define the RUL R_{t,x_t} of a system at time t with observed degradation x_t as the random variable:

$$R_{t,x_t} = \inf\{r > 0 : X_{t+r} \geq L | X_t = x_t\}.$$

The cdf of R_{t,x_t} given the known parameters, will be denoted by $F_R(r|\Theta, X_t = x_t)$. Since the stochastic process is increasing, it can be expressed as:

$$F_R(r|\Theta, X_t = x_t) = P(R_{t,x_t} \leq r | X_t = x_t) = P(X_{t+r} \geq L | X_t = x_t)$$
$$= P(X_{t+r} - X_t \geq L - x_t) = F_{\Theta(r)}(L - x_t). \tag{1}$$

2.2. Predictive Maintenance Decision Rule

Let $\{T_n\}_{n \in \mathbb{N}}$ be the aperiodic sequence of inspection times ($T_0 = 0$). At each inspection, one must take the required maintenance decision according to the condition of the system at that

time. To avoid the occurrence of system failure, a preventive threshold $M < L$ is chosen such that preventive action is taken when the degradation level crosses this threshold. The value of M is a decision variable to be optimized. We assume that the maintenance actions are performed in a negligible time and T_n^- refers to the time just before the maintenance. The possible scenarios which can arise are:

- If $X_{T_n^-} \geq L$, the system has failed and is correctively replaced.

- If $M \leq X_{T_n^-} < L$, the system has not yet failed but it has deteriorated to the extent that it is considered as worn out. In this case a preventive action is taken.

- If $X_{T_n^-} < M$, the system is still properly functioning and there is no need for replacement. The system is left as it is.

Both preventive and corrective replacements are perfect and reset the system to "as good as new". Hence after the inspection we have:

$$
X_{T_n} = \begin{cases} 0 & \text{if } X_{T_n^-} \geq M, \\ X_{T_n^-} & \text{if } X_{T_n^-} < M. \end{cases}
$$

In all cases, the inspection scheduling is carried out based on the RUL which is defined as the remaining duration for which the system will work before it fails. The main idea of an RUL based inspection plan is that the next inspection time is chosen so that the probability of failure before the inspection remains lower than a value p $(0 < p < 1)$. Examples of such inspection plannings can be found in [16] and more recently in [5]. For this reason, let us define the p-quantile of the RUL distribution:

$$
\tau_p(X_{T_n}) = \{\triangle t : \Pr(X_{T_n + \triangle t} \geq L | X_{T_n}) = p\} = \{\triangle t : F_R(\triangle t | \Theta(\triangle t); X_{T_n}) = p\}. \tag{2}
$$

Then the next inspection time is given by

$$
T_{n+1} = T_n + \tau_p(X_{T_n}).
$$

This inspection plan provides a reliability (safety) level equal to $(1 - p)$ in which p is a decision variable to be jointly optimized with M. We remark that periodic inspections can be considered by replacing $\tau_p(X_{T_n})$ by a constant throughout the paper. In that case, this constant becomes a decision variable instead of p.

2.3. Cost Function

The inspections are planned sequentially and each of them incurs a cost C_i. At each inspection, a preventive or corrective action is performed if necessary, with costs C_p and C_c respectively. Clearly, $C_c > C_p$. Moreover, since failure can only be detected through inspection, there is a system downtime after failure and an additional cost at a rate of C_d is incurred from the failure time until the next replacement time. Hence the cumulative maintenance cost is:

$$C(t) = C_i N_i(t) + C_p N_p(t) + C_c N_c(t) + C_d d(t),$$

where $N_i(t)$, $N_p(t)$, and $N_c(t)$ are respectively the number of inspections, the number of preventive replacements, and the number of corrective replacements in $[0, t]$. Furthermore, $d(t)$ is the total time passed in a failed state in $[0, t)$. In other words:

$$N_i(t) = \sum_{n \in \mathbb{N}} I_{\{t \geq T_n\}}, \qquad N_p(t) = \sum_{n \in \mathbb{N}} I_{\{t \geq T_n\}} I_{\{M \leq X_{T_n^-} < L\}},$$

$$N_c(t) = \sum_{n \in \mathbb{N}} I_{\{t \geq T_n\}} I_{\{X_{T_n^-} \geq L\}}, \qquad d(t) = \int_0^t I_{\{X_s \geq L\}} ds.$$

The ratio of the expected cost on one regenerative cycle over the expected length of the regenerative cycle will be considered in the following to determine the decision variables. Let us define it as:

$$EC = \frac{E[C(S_1)]}{E[S_1]},$$

where S_1 is the length of one renewal cycle i.e. the time between two successive replacements. From classical renewal theory, it is well known that if the degradation process has the regenerative property, EC is equal to the long-run average cost per time unit EC_∞ for almost any realization of the process, i.e.,

$$EC = \frac{E[C(S_1)]}{E[S_1]} = \lim_{t \to \infty} \frac{E[C(t)]}{t}.$$

Expectations over a renewal cycle are tricky in case of aperiodic inspections. Hence, EC is obtained from EC_∞ by considering Markov renewal properties as explained in next section.

3. Static Aperiodic Maintenance Policy: System State and Long Run Maintenance Cost

In this section, we characterize the evolution of the maintained system at steady state and derive an expression for the long run expected maintenance cost. The considered maintenance policy is

based on the maintenance decision rule given in Section 2.2. It is qualified as "static" which means that the decision variables p and M do not change over time. They are determined in order to minimize the cost ratio EC. The following subsection is related to the classical case with known degradation parameters. The extension to unknown degradation parameters is given in the second subsection.

3.1. The Case of Known Degradation Parameters

In this subsection, we discuss the case in which the vector of parameters Θ of the degradation process is known. Let $\{Y_n = X_{T_n}\}_{n \in \mathbb{N}}$ be the discrete-time random process describing the system state at each inspection time. Grall et al. [10] derived the properties of the process $\{X_t\}_{t \geq 0}$ and the embedded chain $\{Y_n\}_{n \in \mathbb{N}}$, when $\{X_t\}_{t \geq 0}$ is a gamma process. These properties are extended and listed for any monotone increasing stochastic process as follows. We consider the time between inspections to be based on the p-quantile of the system's remaining useful life, i.e., on $\tau_p(\cdot)$ (see (2)). It is worthwhile mentioning that the case of periodic inspection can be obtained by replacing the function τ_p by a constant. All the calculations remain the same.

- The process $\{X_t\}_{t \geq 0}$ is a regenerative process with regeneration times being the dates of replacement. Also, it is a semi-regenerative process with semi-regeneration times being the inspection times.

- The discrete time process $\{Y_n\}_{n \in \mathbb{N}}$ is a Markov chain with continuous state space $[0, M)$ and if δ_0 denotes the Dirac delta function, the transition probability density function (conditioned on the current state) is:

$$\Pr(\mathrm{d}y | x, \Theta) = \overline{F}_{\Theta(\tau_p(x))}(M - x)\delta_0(\mathrm{d}y) + f_{\Theta(\tau_p(x))}(y - x)I_{\{x \leq y < M\}}\mathrm{d}y, \tag{3}$$

where

$$\overline{F}_{\Theta(\tau_p(x))}(y) = \int_y^{+\infty} f_{\Theta(\tau_p(x))}(u)\mathrm{d}u.$$

The probability function $\Pr(\mathrm{d}y | x, \Theta)$ describes the probability of transition from state x to state y. The mass part is related to the degradation level $y = 0$ which is obtained after maintenance. The density part describes the degradation in the absence of maintenance.

- The Markov chain $\{Y_n\}_{n \in \mathbb{N}}$ is Harris ergodic with a unique stationary probability distribution π_Θ:

$$\pi_\Theta(\mathrm{d}x) = a_\Theta \delta_0(\mathrm{d}x) + (1 - a_\Theta)b_\Theta(x)\mathrm{d}x \tag{4}$$

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with

$$a_\Theta = \frac{1}{1 + \int_0^M B_\Theta(x)\mathrm{d}x}, \qquad b_\Theta(y) = \frac{a_\Theta}{1 - a_\Theta} B_\Theta(y),$$

where

$$B_\Theta(y) = f_{\Theta(\tau_p(0))}(y) + \int_0^y B_\Theta(x) f_{\Theta(\tau_p(x))}(y - x)\mathrm{d}x.$$

- The following result follows from the fact that $\{X_t\}_{t\geq 0}$ is a semi-regenerative process and the unique stationary probability distribution π_Θ exists for the embedded Markov chain $\{Y_n\}_{n\in\mathbb{N}}$. We have:

$$EC = \frac{E[C(S_1)]}{E[S_1]} = EC_\infty = \frac{E_{\pi_\Theta}[C(T_1)]}{E_{\pi_\Theta}[T_1]},$$

where T_1 is the first inspection time. This means that an expression for the maintenance cost EC, as well as EC_∞, can be obtained from the analysis of the system behavior on a semi-regenerative cycle (i.e. between two inspections) instead of a regenerative cycle (i.e. between two replacements).

Therefore, we have:

$$EC = \frac{C_i E_{\pi_\Theta}[N_i(T_1)]}{E_{\pi_\Theta}[T_1]} + \frac{C_p E_{\pi_\Theta}[N_p(T_1)]}{E_{\pi_\Theta}[T_1]} + \frac{C_c E_{\pi_\Theta}[N_c(T_1)]}{E_{\pi_\Theta}[T_1]} + \frac{C_d E_{\pi_\Theta}[d(T_1)]}{E_{\pi_\Theta}[T_1]}. \qquad (5)$$

Obviously, $E_{\pi_\Theta}[N_i(T_1)] = 1$. The other quantities, $E_{\pi_\Theta}[N_p(T_1)]$, $E_{\pi_\Theta}[N_c(T_1)]$, $E_{\pi_\Theta}[d(T_1)]$, and $E_{\pi_\Theta}[T_1]$, are computed by integration with respect to π_Θ as follows:

$$E_{\pi_\Theta}[N_p(T_1)] = P_{\pi_\Theta}(M \leq X_{T_1^-} < L)$$
$$= \int_0^M \left[\overline{F}_{\Theta(\tau_p(x))}(M - x) - \overline{F}_{\Theta(\tau_p(x))}(L - x) \right] \pi_\Theta(\mathrm{d}x),$$

$$E_{\pi_\Theta}[N_c(T_1)] = P_{\pi_\Theta}(X_{T_1^-} \geq L) = \int_0^M \overline{F}_{\Theta(\tau_p(x))}(L - x)\pi_\Theta(\mathrm{d}x),$$

$$E_{\pi_\Theta}[d(T_1)] = \int_0^M \left[\int_0^{\tau_p(x)} \overline{F}_{\Theta(\tau_p(x))}(L - x)\mathrm{d}s \right] \pi_\Theta(\mathrm{d}x),$$

$$E_{\pi_\Theta}[T_1] = \int_0^M \tau_p(x)\pi_\Theta(\mathrm{d}x).$$

3.2. The Case of Unknown Degradation Parameters

In practice, the vector of parameters Θ of the degradation process is not known. In this paper, we propose to use the available information about the parameters as a prior distribution, then update

this information using Bayesian methods following the inspection planning which is given by the decision rule. Let us first define the prior distribution of Θ which depends on hyperparameters Ξ. It is denoted by g_Ξ hereafter.

The distribution of the residual life of the system is obtained from g_Ξ and employed to assess the next inspection time. Knowing the prior distribution of Θ, the cdf of remaining useful life can be obtained by:

$$F_R(r|X_t = x_t) = \int F_{R|\Theta}(r|\Theta, X_t = x_t)g_\Xi(\Theta)\mathrm{d}\Theta. \tag{6}$$

For unknown parameters and given the prior distribution, the four properties of the process $\{X_t\}_{t\geq0}$ and of the chain $\{Y_n = X_{T_n}\}_{n\in\mathbb{N}}$ mentioned in Section 3.1 can be restated as follows.

- The process $\{X_t\}_{t\geq0}$ is a regenerative and semi-regenerative process with regeneration times and semi-regeneration times, S_1 and T_1, respectively.

- The process $\{Y_n\}_{n\in\mathbb{N}}$ is a Markov chain that takes values in $[0, M)$ and has transition probability density function (conditioned on the current state)

$$\Pr(\mathrm{d}y|x) = \overline{F^*}_{\tau_p(x)}(M - x)\delta_0(\mathrm{d}y) + f^*_{\tau_p(x)}(y - x)I_{\{x\leq y<M\}}\mathrm{d}y, \tag{7}$$

  where

$$f^*_{\tau_p(x)}(z) = \int f_{\Theta(\tau_p(x))}(z)\mathrm{d}\Theta, \tag{8}$$

  and

$$\overline{F^*}_{\tau_p(x)}(z) = \int_z^\infty f^*_{\tau_p(x)}(u)du.$$

- Similarly, the unique stationary probability distribution of $\{Y_n\}_{n\in\mathbb{N}}$ is given by:

$$\pi(\mathrm{d}x) = a\delta_0(\mathrm{d}x) + (1 - a)b(x)\mathrm{d}x, \tag{9}$$

  with

$$a = \frac{1}{1 + \int_0^M B(x)\mathrm{d}x}, \qquad b(y) = \frac{a}{1 - a}B(y),$$

  where

$$B(y) = f^*_{\tau_p(x)}(y) + \int_0^y B(x)f^*_{\tau_p(x)}(y - x)\mathrm{d}x.$$

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- As for the case with known degradation parameters, we find:

$$EC = EC_\infty = \frac{E_\pi[C(T_1)]}{E_\pi[T_1]}. \tag{10}$$

The expectations for the different costs are computed as follows:

$$E_\pi(N_p(T_1)) = P_\pi(M \le X_{T_1^-} < L)$$
$$= \int_0^M \left[\overline{F^*}_{\tau_p(x)}(M - x) - \overline{F^*}_{\tau_p(x)}(L - x) \right] \pi(\mathrm{d}x),$$

$$E_\pi(N_c(T_1)) = P_\pi(X_{T_1^-} \ge L) = \int_0^M \overline{F^*}_{\tau_p(x)}(L - x)\pi(\mathrm{d}x),$$

$$E_\pi(d(T_1)) = \int_0^M \left[\int_0^{\tau_p(x)} \overline{F^*}_s(L - x)\mathrm{d}s \right] \pi(\mathrm{d}x),$$

$$E_\pi(T_1) = \int_0^M \tau_p(x)\pi(\mathrm{d}x).$$

3.3. Optimization of The Maintenance Policy

The choice of the maintenance threshold value and inspection dates will obviously influence the economic performance of the maintenance policy. Thus, the objective is to obtain values of the decision variables p and M such that the maintenance cost EC reaches its minimum value. For the static aperiodic maintenance cases described in Sections 3.1 and 3.2 respectively, the optimal decision variables, p^* and M^*, are such that:

$$EC(p^*, M^*) \le EC(p, M), \qquad \forall (p, M) \in [0, 1] \times [0, L],$$

where $EC(p, M)$ is as in (5) or (10), respectively. Numerical illustrations of this procedure are given in Section 6.

4. From Static to Adaptive Predictive Aperiodic Maintenance Policy

The previous section gives probabilistic characteristics of the system evolution under a given aperiodic maintenance policy. It allows us to derive a cost rate EC and hence optimize the decision variables p and M, which are presumed to be constant over time. In that case, the cost EC is equal to the long run expected maintenance cost per time unit. Section 4.1 describes a procedure which updates the estimates of the model parameters using Bayesian methods and adjusts the maintenance decision rule accordingly. In that case, the cost EC which is obtained at a given cycle

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is related to current level of information and it changes with the change of model parameters. As a consequence EC is no longer equal to EC_∞. We heuristically propose to use the cost ratio EC as a criterion to determine the decision variables according to the current level of knowledge on the degradation process. Three alternative maintenance policies are introduced and their properties are derived in Section 4.2.

4.1. Degradation Model Update and Adaptive Maintenance Decision Rule

This Bayesian procedure allows us to update current information about Θ with a new observation. Updating can take place at each inspection or at the end of each cycle with the corresponding number of inspections. Bayesian updates can be used directly to modify the inspection scheduling function. The algorithm is as follows:

● **Update after each inspection.** In this case, only one new observation is available. For a given p and M we have:

Step 1: Initialize the hyperparameters $\Xi_{(0)}$. Furthermore, set $X_0 = 0$, $T_0 = 0$, and $i = 0$.

Step 2: Using (6), evaluate $\triangle t = \tau_p(X_{T_i})$ according to the RUL distribution.

Step 3: Monitor the system at $T_{i+1} = T_i + \triangle t$.

Step 4: Using Bayes' theorem, obtain the joint posterior distribution of Θ with the information available (i.e., X_{T_i} and $X_{T_{i+1}}$) to obtain the new hyperparameters $\Xi_{(i+1)}$.

Step 5: If $X_{T_{i+1}} \geq M$, i.e. the system has failed or is too deteriorated, set $X_{T_{i+1}} = 0$ (corrective or preventive replacement, whichever is needed).

Step 6: Set $i = i + 1$ and then return to Step 2.

● **Update after each cycle.** The only difference with the update after each inspection rule is that computation of the posterior distribution of Θ is postponed until the end of each cycle and the hyperparameters remain unchanged during a specific cycle. Here, the number of the new observations available at each update is equal to the number of inspections in that cycle.

4.2. Static and Dynamic Auto-adaptive Maintenance Policy

The maintenance decision rule described in Section 2.2 depends on two decision variables which are the value of p for the p-quantile of the RUL and the preventive threshold M. As stated in the previous section, the distribution of model parameters and hence the RUL distribution and the expression of the time to the next inspection can be updated without changing the decision variables. In this section, we consider options for maintenance policies as the values of the decision variables or the distribution of the parameters are modified.

Depending on decision variables, policies can be classified in two groups:

Static strategy: The decision variables are fixed over the time horizon in which the policy is implemented.

This strategy can be implemented when the degradation parameters are known or unknown, based on the results in Section 3. For known degradation parameters the expression of τ_p is obtained according to the RUL function given in (1). For unknown degradation parameters the p-quantile of the RUL distribution, given in (6), can remain constant based on prior information or be updated as explained in Section 4.1. The corresponding maintenance policies are respectively called "static non-adaptive" or "static auto-adaptive".

Dynamic strategy: The decision variables can be modified "on-line" and be updated from information which is collected at inspection times.

The resulting policy, which is implemented in the case of unknown degradation parameters, is called a "dynamic auto-adaptive" policy.

To summarize, three different aperiodic predictive maintenance policies can be implemented:

(i) a Static Non-adaptive Aperiodic Predictive (SNAP) policy with fixed decision variables and degradation parameters or hyperparameters,

(ii) a Static Auto-adaptive Aperiodic Predictive (SAAP) policy where only the parameters' distribution (hence the RUL distribution) is updated,

(iii) the Dynamic Auto-adaptive Predictive Aperiodic Maintenance (DAPAM) policy where the parameters' distribution and the decision variables are updated simultaneously.

In a general case, the derivation of optimal decision variables p and M with degradation parameters updates is a difficult problem. Consequently a generic process is proposed which is based on

minimization of the maintenance cost EC given the current available information. This cost can be obtained based on analytical calculations as described in Section 3, The specific proposed rules are as follows:

- For SNAP policy implementation, the fixed decision variables are directly obtained by minimizing EC as derived respectively in Section 3.1 for known degradation parameters and in Section 3.2 for unknown parameters. In the second case the prior distribution of parameters Θ is with initial values of the hyperparameters. In that configuration, EC is equal to the long-run cost rate EC_∞.

- For implementation of the DAPAM policy a heuristic suboptimal procedure is proposed in which the decision variables are optimized for each cycle according to the available knowledge at the beginning of the cycle. The resulting values are used for one renewal cycle, i.e. until the next update. The optimization is based on the cost rate EC as given in Section 3.2 and is conducted using the most recent updated values of the hyperparameters.

Concerning the SAAP policy, the decision variables are fixed. They should be obtained at the initial time, before any update. This policy is considered only for numerical reference in Section 6. Monte Carlo simulation is performed with degradation levels obtained from real values of parameters to estimate the maintenance cost rate.

In the next sections, the precise implementation is given specifically for the inverse Gaussian process. Then the policies are analyzed and compared numerically using the mean estimate of EC as a criterion in order to assess the updating processes and the heuristic choice of decision variables. This estimate is obtained on a finite time horizon.

5. Case of the Inverse Gaussian process

As indicated in the introduction, different stochastic processes can be considered, for example the gamma process and the IG process. Both have the same meaningful physical interpretation for modeling degradation as a limit of a compound Poisson process [26]. In addition these two processes have a high capacity to incorporate random effects and covariates to capture heterogeneities in degradation data. Albeit, the IG process is more appropriate for incorporating prior information, especially conjugate information, into Bayesian inference due to its relationship with the Gaussian

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distribution family. Explicit forms of important related characteristics such as reliability function, quantiles, and the distribution function of the remaining useful lifetime are also easily obtained. Therefore, the IG process is specifically considered in this section for the implementation of the proposed adaptive policies.

For this process, each increment follows an IG distribution, that is, $X_t - X_s \sim IG(\delta^{-1}.(t - s), \lambda.(t - s)^2)$ for all $t > s$. The related pdf and cdf are defined for $x > 0$ as follows:

$$f_{\Theta(t)}(x) = f_{\delta^{-1}t, \lambda t^2}(x) = \sqrt{\frac{\lambda t^2}{2\pi x^3}} \exp\{-\frac{\lambda}{2x}(\delta x - t)^2\},$$

$$F_{\Theta(t)}(x) = F_{\delta^{-1}t, \lambda t^2}(x) = \Phi\left(\frac{\sqrt{\lambda}(t - \delta x)}{\sqrt{x}}\right) - \exp(2t\lambda\delta)\Phi\left(-\frac{\sqrt{\lambda}(t + \delta x)}{\sqrt{x}}\right)$$

where $\delta, \lambda > 0$ and Φ is the cdf of the standard normal distribution. Then, the mean and variance of X_t are t/δ and $t/(\delta^3\lambda)$, respectively. From (1), the cdf of R_{t,x_t} given the known parameters, can be expressed as:

$$F_{R|\delta, \lambda}(r|\delta, \lambda, X_t = x_t) = \Phi\left(\frac{\sqrt{\lambda}(r - \delta(L - x_t))}{\sqrt{L - x_t}}\right) - \exp(2r\lambda\delta)\Phi\left(-\frac{\sqrt{\lambda}(r + \delta(L - x_t))}{\sqrt{L - x_t}}\right).$$

With these considerations, the functions in (3), (4) and (5) can easily be obtained.

In the case of unknown parameters and for Bayesian inference, conjugate priors are employed to ease calculations. Hence, suppose λ has the gamma density function

$$f(\lambda) = \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} \exp\{-\lambda/\beta\}, \tag{11}$$

and let δ have the conditional normal density function with mean ξ and variance σ^2/λ:

$$f(\delta|\lambda) = \sqrt{\frac{\lambda}{2\pi\sigma^2}} \exp\{-\frac{\lambda(\delta - \xi)^2}{2\sigma^2}\}. \tag{12}$$

Then the joint prior distribution of (δ, λ) is given by $f(\delta, \lambda) = f(\delta|\lambda)f(\lambda)$. With this definition, some characteristics of δ and λ are

$$E(\lambda) = \alpha\beta, \ \text{Var}(\lambda) = \alpha\beta^2, \ E(\delta) = \xi, \ \text{and Var}(\delta) = \frac{\sigma^2}{(\alpha - 1)\beta}.$$

Clearly, the slope of an irreversible degradation process must be positive. Hence δ can only take positive real values. To avoid the possibility of obtaining negative degradation slopes, we suppose that $P(\delta \leq 0)$ is negligible (Lu and Meeker [18], and Peng [20]). With a calculation similar to that in Peng [20], (6) and (8) are as follows:

$$F_R(r|X_t = x_t) = 1 - \sqrt{\frac{\beta}{2\pi}} \frac{\Gamma(\alpha + 1/2)r}{\Gamma(\alpha)} \int_0^{L-x_t} z^{-3/2}(\sigma^2 z + 1)^{-1/2}\left(1 + \frac{\beta(\xi z - r)^2}{2z(\sigma^2 z + 1)}\right)^{-(\alpha+1/2)} dz,$$

and

$$f^*_{\tau_p(x)}(z) = \sqrt{\frac{\beta}{2\pi}} \frac{\Gamma(\alpha+1/2)\tau_p(x)}{\Gamma(\alpha)} z^{-3/2}(\sigma^2 z + 1)^{-1/2}\left(1 + \frac{\beta(\xi z - \tau_p(x))^2}{2z(\sigma^2 z + 1)}\right)^{-(\alpha+1/2)}.$$

Then, functions in (7), (9) and (10) can be obtained. The posterior functions of (δ, λ) needed in Section 4.1 are also given in the Appendix.

6. Numerical Study

A simulation study is conducted in the case of an IG process for better insight through numerical illustration. The aim of the numerical experiments is twofold.

i) To assess the performance of the proposed auto adaptive maintenance policies and compare them with non-adaptive policies.

ii) To study the behavior of the decision variables and model parameters and highlight the effect of the Bayesian updates.

6.1. Simulation Framework

Samples of degradation increments are generated from an IG process with fixed parameters δ_{real} and λ_{real}. Three values of the mean degradation rate (0.6, 1 and 2) and three values of the variance (0.25, 1 and 4) are considered. The nine resulting configurations are summarized in Table 1. The failure level is set at $L = 9$ for all samples. In the case of unknown parameters a prior distribution is assumed as described in Section 5. The initial values considered for hyperparameters are as follows:

$$\alpha = 1.5, \ \beta = \frac{2}{3}, \ \xi = 1, \text{ and } \sigma = \frac{1}{\sqrt{3}}. \tag{13}$$

Figure 1 displays the graph of the corresponding joint pdf of the degradation parameters. The mean and variance of the degradation parameters for this first prior are respectively

$$E[\lambda] = E[\delta] = 1, \ \text{Var}[\delta] = 1, \text{ and } \text{Var}[\lambda] \simeq 0.67.$$

The hyperparameter relationships have the following interpretations: as $E[\delta] = \xi$, the mean degradation rate is increasing as ξ decreases. The variance of the degradation rate increases as λ decreases hence when the product $\alpha.\beta$ of the hyperparameters decreases. The variance of δ (respectively λ)

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increases with σ (respectively $\alpha.\beta^2$). A second example of a prior distribution with more uncertainty about degradation parameters will be considered in Section 6.2.3 so as to analyse sensitivity to choice of prior. The value of σ will be significantly increased with the second prior.

Table 1: Different configurations for parameters of degradation process.

Config. nb.	δ_{real}	λ_{real}	Variance
1	1.667	0.864	0.25
2	1.667	0.216	1
3	1.667	0.054	4
4	1	4	0.25
5	1	1	1
6	1	0.25	4
7	0.5	32	0.25
8	0.5	8	1
9	0.5	2	4

6.2. Comparison of Maintenance policies

In order to assess and compare the different maintenance policies, hence the relevance and responsiveness of the dynamic auto-adaptive process, the behavior of the system is considered for a limited time horizon (0 to $T_{\text{end}} = 200$). The criterion for comparison is the maintenance cost estimated on the chosen time horizon. Classical Monte Carlo simulation is considered for estimation: the maintenance decision rules are applied on simulated degradation paths. For the DAPAM policy, the optimization for decision variables is conducted by a classical gradient-based optimization algorithm and requires the most computational effort. With R software on an Intel core i5 processor the time required to calculate a cost value (stationary law + cost assessment) is about 30 seconds. Around 30 cost evaluations may be necessary for convergence of one optimization.

6.2.1. Static non-adaptive (SNAP) maintenance policy

Let consider first the case of static maintenance policy without Bayesian update. The two options with known and unknown degradation parameters are investigated hereafter. In both cases the decision variables are constant and optimized from information available at time $t = 0$.
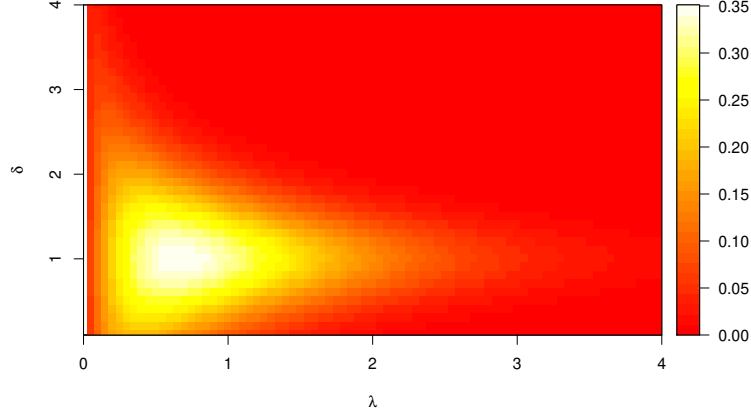
Figure 1: The joint density function of (λ, δ) for $\alpha = 1.5$, $\beta = \frac{2}{3}$, $\xi = 1$, and $\sigma = \frac{1}{\sqrt{3}}$.

● **Known degradation parameters**

The maintenance decision rule is derived and tuned assuming that the degradation follows an IG process with known parameters. The values p^* and M^* of the decision variables are those minimizing EC in (5). These values are given in the last three rows of Table 2 for each configuration. Rows 2 and 3 correspond to the values of the maintenance cost estimated on a finite time horizon $\widehat{EC}$ and its standard deviation. The estimation is based on simulation of 45 samples of maintained system degradation paths over a finite time horizon.

It can be seen in Table 2 that the values of theoretical and estimated maintenance cost rates, i.e. $\widehat{EC}$ and EC, are close to each other. Moreover, the standard deviations of the estimations are very small. In other words, the decision variables optimized with respect to $\widehat{EC}$ are relevant with estimation on a finite time horizon.

● **Unknown degradation parameters**

Here, an IG process is assumed with the proposed prior distribution of hyperparameters given in (13). The initial values of the hyperparameters are not updated. Analytical calculations from Section 3.2 are used to obtain and minimize EC. The optimal decision variables and the corresponding minimum cost obtained by numerical methods are respectively:

$$p^* = 0.115, \ M^* = 6.49, \text{ and } EC^* = 1.19. \tag{14}$$

Table 2: The optimal decision variables and costs for the SNAP policy when the parameters are known.

Config. nb.	1	2	3	4	5	6	7	8	9
$\widehat{EC}$	0.386	0.472	0.527	0.605	0.739	0.827	1.141	1.323	1.540
SD	0.047	0.074	0.108	0.047	0.087	0.129	0.057	0.088	0.141
EC^*	0.394	0.499	0.549	0.609	0.739	0.856	1.136	1.313	1.552
p^*	0.017	0.033	0.046	0.015	0.033	0.056	0.012	0.028	0.061
M^*	7.05	6.28	6.01	7.38	6.59	6.21	7.74	7.02	6.37

The values of p and M given in (14) can be used as decision variables to apply the SNAP policy to the configurations described in Table 1. The related estimated maintenance costs on the finite time horizon are given in Table 3. As expected, they are higher than for the case of known parameters. However, the policy seems to be robust.

Table 3: Values of $\widehat{EC}$ and their standard deviations for the SNAP policy assessed based on the hyperparameters given in (13).

Config. nb.	1	2	3	4	5	6	7	8	9
$\widehat{EC}$	0.462	0.530	0.559	0.718	0.791	0.869	1.365	1.450	1.579
SD	0.098	0.130	0.152	0.105	0.122	0.154	0.129	0.136	0.164

6.2.2. Auto-adaptive maintenance policies

Here, we assume that the degradation parameters are unknown and consider policies involving Bayesian updates. First, the case with constant decision variables (the SAAP maintenance policy) is considered, and then the case where the decision variables are automatically adapted to the update of the hyperparameters (the DAPAM policy) is considered.

● **Static auto-adaptive (SAAP) maintenance policy**

The hyperparameters are updated at each inspection time. The expression for the p-quantile of the RUL is modified accordingly but the values of p and M remain constant over the whole time horizon. The maintenance cost $\widehat{EC}$ depends on the characteristics of the Bayesian update sequence up to T_{end} and these are related to the real degradation parameters. Hence the optimization of the

decision variables requires knowledge of the real degradation parameters. These parameters are not available in reality but numerical experiments are achievable. Table 4 gives the minimum estimated values of the cost rate over a finite horizon for the SAAP policy with updates at each inspection time. A comparison of the values in Table 4 with the values in the last three rows in Table 2 reveals the efficiency of the Bayesian update in the maintenance decision rule. The minimum values of $\widehat{EC}$ are very close to those obtained for known decision variables, which means that the influence of the prior weakens over time, and that the Bayesian update is efficient from the maintenance point of view.

Table 4: Values of the optimal estimated maintenance cost over a finite time horizon and corresponding decision variables for the SAAP policy with update at each inspection time.

Config. nb.	1	2	3	4	5	6	7	8	9
$\widehat{EC}^*$	0.394	0.488	0.533	0.612	0.736	0.838	1.144	1.322	1.544
p^*	0.028	0.046	0.062	0.028	0.038	0.064	0.06	0.036	0.07
M^*	7.2	6.4	6.2	7.4	6.4	6.4	7.6	7	6.6

● **Dynamic auto-adaptive (DAPAM) maintenance policy**

The DAPAM policy allows decision variables to be optimized sequentially. The optimization can be performed at any time according to the available information described by the hyperparameters at that time.

The estimation of the maintenance cost on successive renewal cycles requires Monte Carlo simulation of degradation paths followed by optimization of p and M. The optimization involves numerical calculations for the stationary law and long-run cost assessment as described in Section 3.2. The expected maintenance cost on a finite time horizon $\widehat{EC}$ is estimated after the last update when time T_{end} has been reached. The results are given in Table 5. We see that the cost values are close to those of the SNAP policy with known parameters (Table 2) for each configuration. The relative deviations between the costs of the two policies are lower for high degradation rates (e.g $\delta = 0.5$) than for low degradation rates (e.g. $\delta = 1.667$). This is because the number of renewal cycles and hence the number of updates from 0 to T_{end} increases with the mean deterioration speed. The smallest relative difference of costs is obtained for $\delta = 1$ which is the case for which the initial values of hyperparameters fit best. Overall, the results show the promise of the DAPAM policy.

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Table 5: Values of the maintenance cost over a finite time horizon and corresponding decision variables for the DAPAM policy with update at each cycle.

Config. nb.	1	2	3	4	5	6	7	8	9
$\widehat{EC}$	0.375	0.504	0.582	0.614	0.753	0.823	1.190	1.318	1.580
SD	0.035	0.086	0.135	0.043	0.095	0.121	0.038	0.068	0.142
$\bar{p}$ at T_{end}	0.016	0.036	0.050	0.018	0.036	0.058	0.020	0.032	0.068
$\bar{M}$ at T_{end}	7.13	6.17	5.95	7.18	6.46	6.06	7.24	6.82	6.23

6.2.3. Effect of prior knowledge

In the case of unknown degradation parameters, the initial knowledge about the degradation may have an influence on the economic performance of the auto-adaptive policies. In order to illustrate this, let us consider another set of initial values for hyperparameters as follows:

$$\alpha = 3, \ \beta = \frac{2}{3}, \ \xi = \frac{1}{2}, \text{ and } \sigma = 12. \tag{15}$$

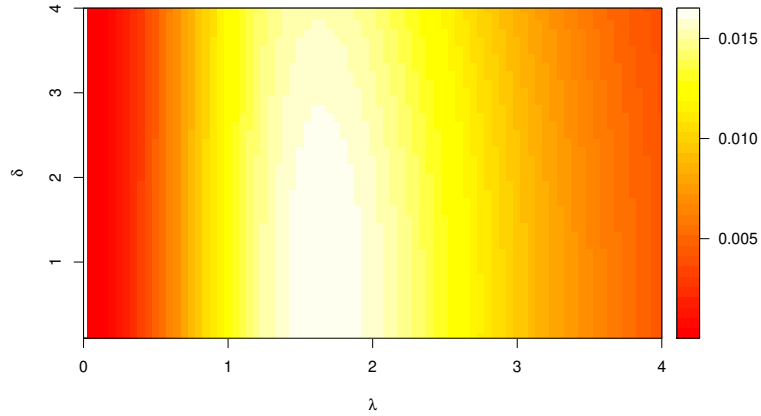The mean and variance of the degradation parameters for this prior are respectively

Figure 2: The joint density function of (λ, δ) for $\alpha = 3$, $\beta = \frac{2}{3}$, $\xi = \frac{1}{2}$, and $\sigma = 12$.

$$E[\delta] = 0.5, \ \text{Var}[\delta] = 108, \ E[\lambda] = 2, \text{ and } \text{Var}[\lambda] \simeq 1.33.$$

The graph of the pdf of (λ, δ) is displayed in Figure 2. For these values of the hyperparameters, the SNAP policy for the optimization of the long term cost rate leads to

$$p^* = 0.603, \ M^* = 6.24, \text{ and } \ EC^* = 2.43. \tag{16}$$

This cost can be compared with the cost in (14) which is significantly lower. Actually, the distribution of degradation parameters from (11) and (12) is less informative with the set of hyperparameters given in (15) than with the set of hyperparameters given in (13) (the variances of δ and λ are greater). It follows that the density function of (λ, δ) is more spread out. The numerical results corresponding to the different policies are given in Table 6 for Configurations 5 and 9. The cost standard deviations for both cases are around 0.15. The cost values for SNAP, SAAP, and DAPAM policies can be compared with those in Tables 3, 4 and 5. The SNAP policy has been implemented assuming unknown parameters (hyperparameters from (15)) and decision variables as in (16). The SAAP policy is implemented for the best case i.e. the decision variables are those minimizing $\widehat{EC}$.

Table 6: Costs of $\widehat{EC}$ for the SNAP, SAAP, and DAPAM policies in Configurations 5 and 9 with the hyperparameters given in Eq. (15). The bracketed values are accumulated costs from the beginning of the second renewal cycle to T_{end}.

Config. nb.	$\widehat{EC}$ (SNAP)	$\widehat{EC}$ (SAAP)	$\widehat{EC}$ (DAPAM)	$\bar{p}$ at T_{end} (DAPAM)	$\bar{M}$ at T_{end} (DAPAM)
5	1.53	0.777	1.02 (0.739)	0.035	6.50
9	3.02	1.57	1.80 (1.53)	0.066	6.25

With initial values of hyperparameters given in (15), the RUL prediction is an over-estimate. This is very disadvantageous when using the DAPAM policy because it leads to a very late first inspection. Hence there is only one inspection in the first renewal cycle which is at $t = 25$. A corrective maintenance is performed at that time after a long period of unavailability. Due to the unsuitable initial prior distribution, the cost incurred during the first renewal cycle is very high and more than 12% of the time horizon elapses without any update. Without the cost of the

first renewal cycle or, in other words, if we start counting after the first update, the cumulated maintenance cost rate (bracketed values in Table 6)) is very close to the cost for the case of known parameters. This shows that the first few Bayesian updates and especially the very first one, are fundamental from the maintenance point of view. At the end of the time horizon, the behavior of the system with the DAPAM policy is very close to the behavior in the case when the degradation parameters are known. We remark that from a practical point of view it can be difficult to choose initial hyperparameters in some cases. If maintenance planners want to avoid failures and start with a conservative point of view it may be interesting to consider a prior distribution with an underestimated value of δ, i.e., with overestimation of the degradation rate. The drawback to this may be a significant increase of maintenance cost for the first cycle.

6.3. Parameters of the adaptive maintenance policy

In this section, we present three typical cases which depict the evolution of the distribution of the unknown parameters during the process of Bayesian updating for systems maintained with the DAPAM policy. With the initial values of hyperparameters as in (13), Configurations 3 and 8 with $\delta = 1.667$ and $\delta = 0.5$ respectively represent examples of low and high degradation rates while Configuration 5 with initial values of hyperparameters as in (15) illustrates the case of a poor prior.

Figures 3, 5 and 7 display the evolution of the means and standard deviations of the decision variables as a function of time for these three cases. For example, Figure 3 shows the evolution of $\bar{p}$ and $\bar{M}$ for Configuration 3. We see that over time, the standard deviations of these estimators decrease. Their values tend to the values of p^* and M^* obtained in the case of known parameters as given in Table 2. We note that in the case of Configuration 5 (Figure 7), the first Bayesian update which is taken around $t = 25$ has a significant impact on the rate of increase of the trend.

Figures 4, 6 and 8 show the evolution of the hyperparameters. The joint distribution of (δ, λ) is such that $E(\delta) = \xi$ and $E(\lambda) = \alpha\beta$. The red lines on the graphs of the evolutions of $\alpha\beta$ and ξ correspond to the values of δ_{real} and λ_{real} in the configuration. These figures show that even though δ and λ are random variables, the means of their distributions, i.e ξ and $\alpha\beta$, tend to the values δ_{real} and λ_{real} with the successive Bayesian updates. At the same time, the variance of λ and δ, i.e. $\alpha\beta^2$ and $\dfrac{\sigma^2}{(\alpha - 1)\beta}$, decrease, reflecting the high certainty of the tendency. This procedure is more obvious in Figure 6. This is due to the higher number of updates which is the consequence of the higher degradation rate in configuration 8. Once again, Figure 8 shows the influence of the

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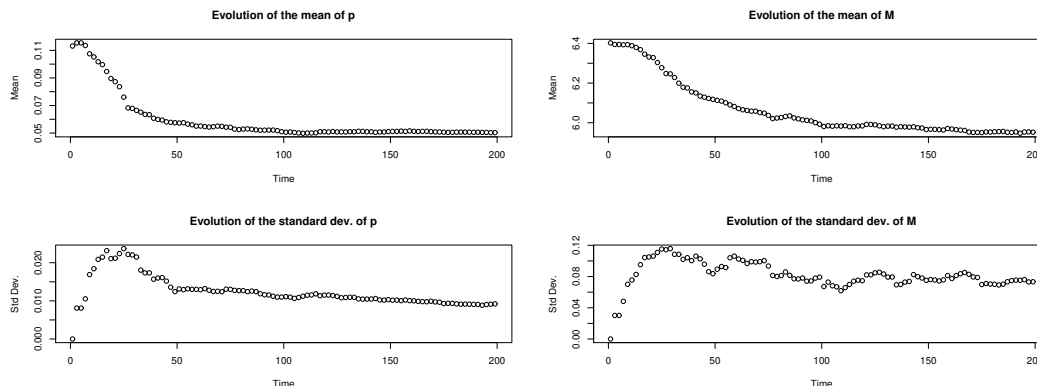Bayesian updates in the convergence even when the priors have not been very well chosen.

Figure 3: The evolution of decision variables p (left) and M (right) for the DAPAM policy in Configuration 3 with $\alpha = 1.5$, $\beta = \frac{2}{3}$, $\xi = 1$, and $\sigma = \frac{1}{\sqrt{3}}$.

7. Conclusion

Maintenance has become an increasingly important area of research and a large body of literature has been developed for it. However, almost all studies share the assumption that the parameters of the degradation process are known. In this paper, we consider a dynamic auto- adaptive predictive maintenance (DAPAM) policy for single-unit systems with unknown degradation parameters, assuming that the degradation is governed by a monotone increasing process and that the unknown parameters jointly follow a prior distribution with specified hyperparameters. The Bayes method is employed to update information about parameters over time. Simulation studies are conducted to illustrate the behavior of the policy and maintenance decision variables are obtained using a mean cost rate between two replacements as a criterion in the case of degradation governed by an IG process. Moreover, different alternative policies have been discussed and it is shown that when compared with the DAPAM policy the latter performs much better when considering its ability to get close to the optimal decision variables obtained in the case that the degradation parameters are known.

The proposed dynamic decision rule is heuristic. In case of unknown or uncertain parameters and for continuous deterioration processes, the optimal policy structure is unknown. Future research
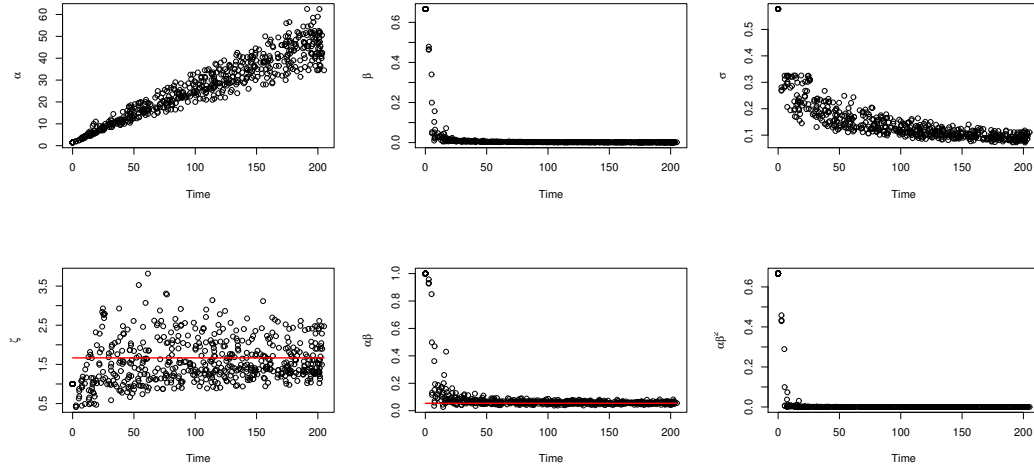
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Figure 4: The evolution of hyperparameters for the DAPAM policy in Configuration 3 with $\alpha = 1.5$, $\beta = \frac{2}{3}$, $\xi = 1$, and $\sigma = \frac{1}{\sqrt{3}}$.
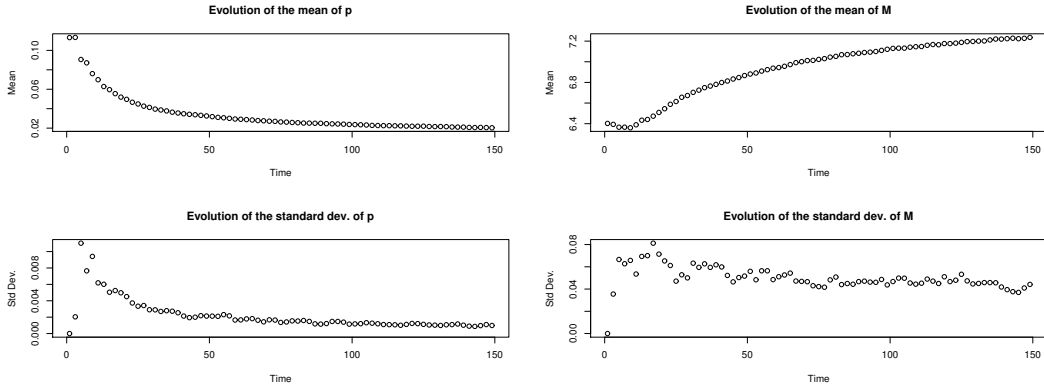

Figure 5: The evolution of decision variables p (left) and M (right) for the DAPAM policy in Configuration 8 with $\alpha = 1.5$, $\beta = \frac{2}{3}$, $\xi = 1$, and $\sigma = \frac{1}{\sqrt{3}}$.

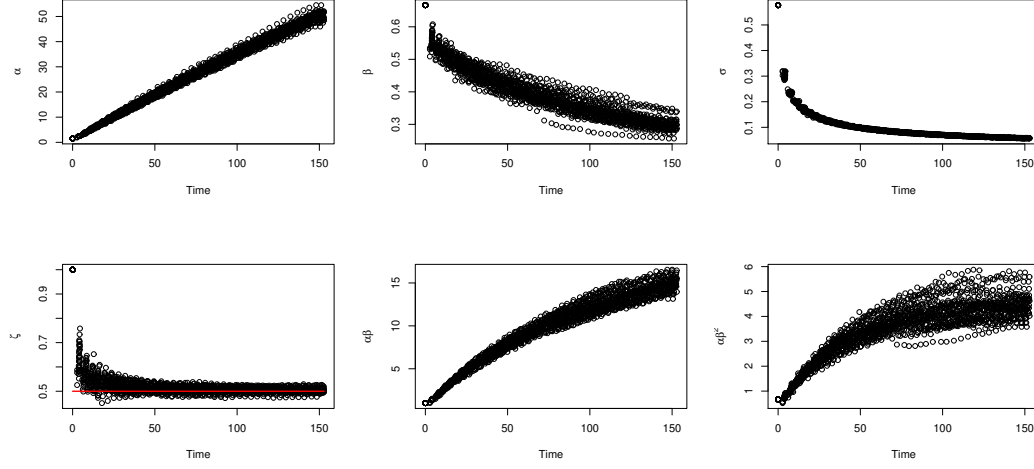Figure 6: The evolution of hyperparameters for the DAPAM policy in Configuration 8 with $\alpha = 1.5$, $\beta = \frac{2}{3}$, $\xi = 1$, and $\sigma = \frac{1}{\sqrt{3}}$.
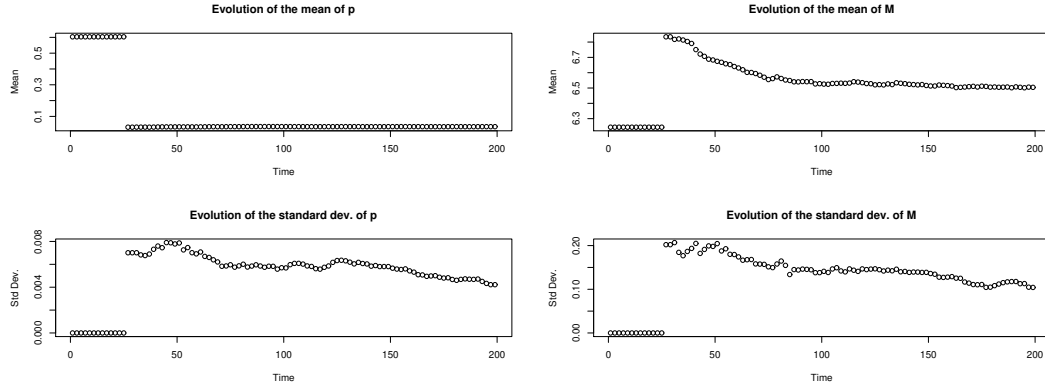

Figure 7: The evolution of decision variables p (left) and M (right) for the DAPAM policy in Configuration 5 with $\alpha = 3$, $\beta = \frac{2}{3}$, $\xi = \frac{1}{2}$, and $\sigma = 12$.
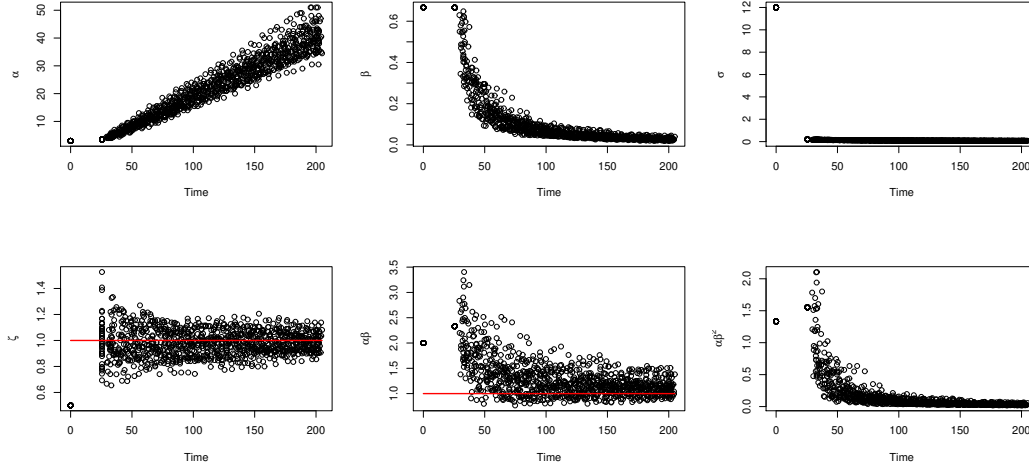
Figure 8: The evolution of hyperparameters for the DAPAM policy in Configuration 5 with $\alpha = 3$, $\beta = \frac{2}{3}$, $\xi = \frac{1}{2}$, and $\sigma = 12$.

needs to be done to characterize properties of the proposed policy in terms of optimality or sub-optimality. One next step would be to obtain analytical results concerning convergence rates of the Bayesian estimation and the long-run cost rate following given numbers of Bayesian updates. Although some advantages of the DAPAM policy are its small number of decision variables and its adaptability to a wide range of situations, a possible weakness could be a slow convergence rate in parameters estimation in certain cases. Actually the considered inspection policy tends to reduce the number of inspections. More numerical analysis needs to be carried out in order to characterize disadvantageous cases. The influences of several things need to be analyzed more deeply including the initial prior distribution, the real degradation process, the inspection cost, and the number of inspections.

There are potential extensions to overcome the possible drawbacks concerning estimation when it is driven by the proposed condition-based inspection policy. An additional rule could be considered to increase the number of inspections when necessary. Frequent periodic inspections could be planned during a given period of time to get more information about degradation and accelerate parameters estimation. In this case a specific additional decision rule needs to be defined, with its additional decision variables, and compared to the proposed one. Natural extensions of the present work could also consider more complex degradation processes with the influence of environmental

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parameters as well as imperfect maintenance actions.

8. Appendix

The appendix gives a proposition which can be used to find the successive updates for parameter distributions in the case of an IG process.

Suppose $X_{t_0}, X_{t_1}, \ldots, X_{t_k}$ are the observed degradation data at times $t_0, t_1, \ldots, t_k$. Then the likelihood function of δ and λ is:

$$L(\delta, \lambda) \propto \lambda^{\frac{k}{2}} \exp\{-\frac{\lambda}{2} \sum_{i=1}^{k} \frac{(\triangle x_i \delta - \triangle t_i)^2}{\triangle x_i}\},$$

where $\triangle x_i = X_{t_i} - X_{t_{i-1}}$ and $\triangle t_i = t_i - t_{i-1}$. Using (11) and (12), The joint posterior distribution of (δ, λ) can be derived as follows:

Proposition: Given the observed data, $Data = \{X_{t_0}, X_{t_1}, \ldots, X_{t_k}\}$, the joint posterior distribution of (δ, λ) is:

$$f(\delta, \lambda | Data) = f(\delta | \lambda, Data) f(\lambda | Data),$$

where

$$f(\delta | \lambda, Data) = \sqrt{\frac{\lambda}{2\pi\sigma^{*2}}} \exp\{-\frac{\lambda(\delta - \xi^*)^2}{2\sigma^{*2}}\},$$

$$f(\lambda | Data) = \frac{\lambda^{\alpha^*-1}}{\Gamma(\alpha^*)\beta^{*\alpha^*}} \exp\{-\lambda/\beta^*\}.$$

The updated hyperparameters are $\xi^* = B/A$, $\sigma^* = A^{-1/2}$, $\alpha^* = \alpha + k/2$, and $\beta^* = (1/\beta + 1/D)^{-1}$; where

$$A = \sum_{i=1}^{k} \triangle x_i + \frac{1}{\sigma^2}, \qquad B = \sum_{i=1}^{k} \triangle t_i + \frac{\xi}{\sigma^2},$$

$$C = \sum_{i=1}^{k} \frac{(\triangle t_i)^2}{\triangle x_i} + \frac{\xi^2}{\sigma^2}, \qquad D = \frac{1}{2}(C - \frac{B^2}{A}).$$

Proof: The joint posterior of the (δ, λ) is

$$f(\delta, \lambda | Data) \propto L(\delta, \lambda) f(\delta | \lambda) f(\lambda)$$

$$\propto \lambda^{\frac{k+1}{2} + \alpha - 1} e^{-\frac{\lambda}{\beta}} \exp\left\{-\frac{\lambda}{2}\left[\frac{(\delta - \xi)^2}{\sigma^2} + \sum_{i=1}^{k} \frac{(\delta \triangle x_i - \triangle t_i)^2}{\triangle x_i}\right]\right\}$$

$$\propto \lambda^{\frac{k+1}{2} + \alpha - 1} e^{-\frac{\lambda}{\beta}} \exp\left\{-\frac{\lambda}{2} A\left[\delta^2 - 2\delta \frac{B}{A} + \frac{C}{A}\right]\right\}$$

$$\propto \lambda^{\frac{k+1}{2} + \alpha - 1} e^{-\frac{\lambda}{\beta}} \exp\left\{-\frac{\lambda}{2} A(\delta - \frac{B}{A})^2 - \frac{\lambda}{D}\right\}$$

$$\propto \lambda^{\frac{1}{2}} e^{-\frac{\lambda}{2} A(\delta - \frac{B}{A})^2} \lambda^{\frac{k}{2} + \alpha - 1} e^{-\lambda[\frac{1}{\beta} + \frac{1}{D}]},$$

where A, B, C, and D are as in the proposition. This implies that the posterior distributions of $\delta | \lambda$ and λ are respectively the normal and gamma distributions with the hyperparameters as given. $\square$

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