Modeling past-dependent partial repairs for condition-based maintenance of continuously deteriorating systems

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Abstract

We are interested in the stochastic modeling of a condition-based maintained system subject to continuous deterioration and maintenance actions such as inspection, partial repair and replacement. The partial repair is assumed dependent on the past in the sense that it cannot bring the system back into a deterioration state better than the one reached at the last repair. Such a past-dependency can affect (*i*) the selection of a type of maintenance actions, (*ii*) the maintenance duration, (*iii*) the deterioration level after a maintenance, and (*iv*) the restarting system deterioration behavior. In this paper, all these effects are jointly considered in an unifying condition-based maintenance model on the basis of restarting deterioration states randomly sampled from a probability distribution truncated by the deterioration levels just before a current repair and just after the last repair/replacement. Using results from the semi-regenerative theory, the long-run maintenance cost rate is analytically derived. Numerous sensitivity studies illustrate the impacts of past-dependent partial repairs on the economic performance of the considered condition-based maintained system.

Keywords: Maintenance, past dependency, partial repair, deterioration process, semi-regenerative process.

Acronyms

ARA_1	arithmetic reduction of age with memory 1
ARD_1	arithmetic reduction of deterioration with memory 1
AGAN	as-good-as-new
CBM	condition-based maintenance
CBMS	condition-based maintained system(s)
CR	corrective replacement(s)
PR	preventive replacement(s)
PPR	preventive partial repair(s)
pdf	probability density function(s)
TBM	time-based maintenance

Notations

X_t	system deterioration level at time t
E_j, S_j	end time of the j-th repair/replacement, starting time of the $(j+1)$-st repair/replacement
$T_{j,k}$	k-th inspection time over the j-th repair/replacement cycle $\left[E_j, E_{j+1}\right)$
$\alpha_0, \alpha\left(X_{E_j^+}\right)$	constant part and $X_{E_j^+}$-dependent part of shape parameter of the deterioration process $\{X_t\}_{E_j^+ \le t \le S_j}$
β	constant scale parameter of $\{X_t\}_{t\ge 0}$
$\Gamma(\cdot), \Gamma(\cdot,\cdot)$	complete Gamma function, upper incomplete Gamma function
$f_{\cdot,\cdot}, \bar{F}_{\cdot,\cdot}$	probability density function, survival function of X_t
L	system failure threshold

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ζ	threshold for triggering a PR or a PPR
η	threshold for choosing between a PR and a PPR
δ	inspection period
λ_0, $\lambda(\cdot,\cdot)$	constant duration for a replacement, additional deterioration-dependent duration for PPR
$g(\cdot \mid \cdot,\cdot)$	pdf of the system deterioration level after a past-dependent PPR
C_m, C_r, C_p, C_c, C_i, C_u	inspection cost, PPR cost, PR cost, CR cost, inactivity cost rate, unavailability cost rate
$C(t)$, C_∞	cumulative maintenance cost up to time t, long-run maintenance cost rate
$N_m(t)$, $N_m\left(\left[0^+, E_1^+\right]\right)$	number of inspections up to time t, and over $\left[0^+, E_1^+\right]$
$N_r(t)$, $N_r\left(\left[0^+, E_1^+\right]\right)$	number of PPR up to time t, and over $\left[0^+, E_1^+\right]$
$N_p(t)$, $N_p\left(\left[0^+, E_1^+\right]\right)$	number of PR up to time t, and over $\left[0^+, E_1^+\right]$
$N_c(t)$, $N_c\left(\left[0^+, E_1^+\right]\right)$	number of CR up to time t, and over $\left[0^+, E_1^+\right]$
$I(t)$, $I\left(\left[0^+, E_1^+\right]\right)$	cumulative duration of the system inactivity up to time t, and over $\left[0^+, E_1^+\right]$
$U(t)$, $U\left(\left[0^+, E_1^+\right]\right)$	cumulative duration of the system unavailability up to time t, and over $\left[0^+, E_1^+\right]$
$\left\{Y_j\right\}_{j\in\mathbb{N}}$	Markov chain describing the system deterioration at repair/replacement times ($Y_j = X_{R_j^+}$)
ΔE_1	length of the first Markov renewal cycle
π	stationary measure of $\left\{Y_j\right\}_{j\in\mathbb{N}}$
$P(\cdot,\cdot)$	transition kernel of $\left\{Y_j\right\}_{j\in\mathbb{N}}$
$E_\pi[\cdot]$	expectation with respect to the measure π

1. Introduction

Maintenance is an effective solution to reduce the system failure, improve the system availability, and extend the system lifetime. It has been adopted in a wide range of systems, such as civil infrastructure (Frangopol and Liu, 2007), manufacturing systems (Lee et al., 2011), automotive vehicles (Lu et al., 2014), I.T. software (Benestad et al., 2009), energy assets (Shafiee and Sørensen, 2017), etc. Maintenance activities comprise perfect and imperfect actions classified following their effects on the condition of maintained systems (Pham and Wang, 1996). If the maintenance recovers the system back to an as-good-as new (AGAN) condition, it is perfect; otherwise, it is imperfect. Typical examples of perfect maintenance are complete replacement and overhaul, and of imperfect maintenance are testing, inspection, minimal repair and partial repair. Unlike restricted applications of perfect maintenance in engineering practice, imperfect maintenance characterizes divers realistic actions whose imperfectness may be caused by various factors such as human errors, spare parts quality, lack of materials, lack of maintenance time, etc. Modeling imperfect maintenance is thus crucial for practical needs.

The present paper deals with a particular kind of imperfect maintenance called *past-dependent partial repair*. It is characterized by the phenomenon that *a partial repair cannot bring a deteriorating system back into a deterioration state better than the one reached at the last repair*. The deterioration paths of draught fans and of gyroscopes provided in (Wang et al., 2018) and (Hu et al., 2018; Pei et al., 2018) are some real-world examples for this phenomenon. Our aim is to develop a condition-based maintenance (CBM) with past-dependent partial repairs for continuously deteriorating systems considering. The main motivation is that CBM is usually more efficient than run-to-failure maintenance and time-based maintenance (TBM) (Ahmad and Kamaruddin, 2012), especially when the system deterioration is nowadays easily accessed thanks to the development of sensor and data transmission technologies (Roy et al., 2016). Notwithstanding, the literature of past-dependent partial repairs is mostly attached to TBM via models with memory (e.g., arithmetic reduction of age (ARA) and arithmetic reduction of intensity (Doyen and Gaudoin, 2004)). Therefore, modeling this kind of imperfect maintenance is still a widely open issue in the contexte of CBM.

In the literature, three main approaches have been employed to take into account the past dependency in imperfect CBM models applied to continuously deteriorating systems.

1. *Repairs number-based modeling*. The first approach considers that the system residual damage after each partial repair exhibits an increasing trend with the sequence of repairs. Since the number of repairs increases over time

until the next replacement, their ability to improve the system deterioration weakens. This leads to the dependency between past and current repairs. Through the repairs number, past dependency effects on the maintained system are modeled. For instance, Liao et al. (2006); Guo et al. (2013); Hu et al. (2018); Pei et al. (2018) have applied this approach to express the past dependency of both the system deterioration level and deterioration rate. Besides, in (Liao et al., 2006; Guo et al., 2013), a higher number of performed repairs also lengthen maintenance duration, but it does not contribute to CBM decision-making. On the contrary, in (Hu et al., 2018; Pei et al., 2018), this number is used as an index to switch between imperfect and prefect maintenance actions, however maintenance durations are omitted.

2. *Virtual age-based modeling*. The second approach links the virtual age of a system to its deterioration level. When a repair removes a portion of virtual age accumulated since the last repair, it also puts the system back to a deterioration level where it was some time before. By this way, the repair is past-dependent. Ahmadi (2014, 2015); Mercier and Castro (2013) are a few authors who employ this approach to developed their CBM models. Using the Kijima's type I model (Kijima, 1989), Ahmadi (2014, 2015) has explained how past-dependent partial repairs affect the system deterioration level and system failure rate. Beyond the impacts on the system deterioration and failure behavior, Mercier and Castro (2013) have based on the ARA with memory 1 (ARA_1) model (Doyen and Gaudoin, 2004) to decide if imperfect repair or perfect replacement is more suitable for a preventive action.
3. *Deterioration level-based modeling*. Unlike the two above approaches, the third one impacts directly the system deterioration. It assumes that a repair just can return the system to a deterioration level worse than the one existing at the last repair. In this spirit, Ponchet et al. (2011) have mimicked the ARA_1 model to build so-called Arithmetic Reduction of Deterioration with memory 1 (ARD_1) model. This model enables a connection to the past in the sense that each repair reduces a part of the deterioration accumulated by the system from the last repair. The ARD_1 model is further implemented by Castro and Mercier (2016) to determine whether a repair or a replacement should be carried out at a preventive maintenance time.

For a better choice among the aforementioned approaches, some comparative works have been done. For instance, Mercier and Castro (2019) have performed stochastic comparisons between the ARA_1 and ARD_1 models under the assumption of a Gamma deteriorating system. Based on a system subject to the Wiener deterioration process, Kahle (2019) has recently compared Kijima's type models (Kijima, 1989) applied for both the system virtual age and the system deterioration. These models also implies that the higher the repairs number, the more the system is deteriorated. Consequently, impacts of repairs number are considered indirectly in the second and third approaches. Moreover, it seems that only the latter is able to take into account directly deterioration information revealed by the system monitoring in repair models.

Owing to its advantages, we have applied the deterioration level-based approach to develop our CBM model considering past-dependent partial repairs for a continuously deteriorating system. Compared to previous works using the same approach (see e.g., (Ponchet et al., 2011; Castro and Mercier, 2016; Zhao et al., 2019)), the developed CBM model has three major differences. Firstly, to express the past dependency of the system deterioration level, we just use a truncated probability distribution. After a repair, the restarting deterioration level of the system is sampled from a probability distribution truncated by the deterioration levels just before a current repair and just after the last repair/replacement. Unlike arithmetic reduction type and Kijima's type models, this simple model allows breaking the memory assumption: *the system after a repair is put back to an exact deterioration level where it was in the past*, which is not easily verified in practice due to the stochastic nature of deterioration process. Secondly, we take into account all the possible the effects of past-dependent partial repairs in an unifying CBM model via restarting deterioration states. There are effects on (*i*) the selection of a type of maintenance actions (either a partial repair or a perfect replacement), (*ii*) the maintenance duration, (*iii*) the deterioration level after a repair, and on (*iv*) the restarting system deterioration behavior. Finally, we analytically derive the long-run expected maintenance cost rate of the condition-based maintained system (CBMS) using the semi-regenerative theory. Even though this approach has now become rather classical in reliability literature (Bérenguer, 2008), its development in the context of past dependency is still meaningful, especially in terms of numerical computation and Monte Carlo simulation.

The remainder of this paper is organized as follows. Section 2 gives a detailed description of the considered CBMS. Sections 3 and 4 are devoted to the mathematical formulation and validation of the asymptotic behavior and the cost model of the maintained system. The sensitivity studies in Section 5 allow us to assess the effects of past-dependent partial repairs on the economic performances of the maintained system. Finally, some conclusions and perspectives are discussed in Section 6.

2. Description of the condition-based maintained system

Let consider a single-unit system subject to continuous deterioration such as wear, fatigue, corrosion, crack growth (Grall et al., 2002). Such a system may consist of one component or one group of associated components whose deterioration state at time $t \geq 0$ can be summarized by a scalar random variable X_t, with $X_0 = 0$. For safety, a high deterioration is unacceptable in engineering practice so that the system is declared as failed whenever its deterioration level exceeds a critical threshold L (i.e., $X_t \geq L$), even if it is physically running. To prevent or correct the system failure, maintenance actions such as inspection, perfect replacement and partial repair are resorted to. The inspection and replacement are assumed memoryless, while the partial repair is past-dependent via the deterioration level given at the last repair. Since these actions are costly, a deterioration-based maintenance policy with two decision thresholds has been proposed to organize them in a proper manner. A threshold on deterioration levels revealed by inspections, denoted $\zeta \in [0, L)$, is used to decide wether or not a preventive intervention should be done at a given inspection time. Another threshold on deterioration levels returned by last repairs, denoted $\eta \in [0, L]$, is used to select the type of a preventive action (i.e., partial repair or perfect replacement). Therefore, let $E_0, E_1, \ldots, E_j, E_{j+1}, \ldots,$ be the successive end-of-repair/replacement times, with $E_0 = 0$, the evolution of the maintained system on the cycle $\left[E_j, E_{j+1}\right)$, $j \in \mathbb{N}$, is as follows.

1. The system is regularly inspected at times $T_{j,k} = E_j + k \cdot \delta$, with $k = 1, 2, \ldots$, until $X_{T_{j,k}} \geq \zeta$. Let S_j denote the starting time of a maintenance action on $\left[E_j, E_{j+1}\right)$, $j \in \mathbb{N}$, then over $\left[E_j, S_j\right]$, X_t acts like a homogeneous Gamma process (Van Noortwijk, 2009) with shape parameter $\alpha_0 + \alpha\left(X_{E_j^+}\right)$ and scale parameter $\beta > 0$

$$\{X_t\}_{E_j^+ \leq t \leq S_j} \sim \mathrm{HGP}\left(\alpha_0 + \alpha\left(X_{E_j^+}\right), \beta\right). \tag{1}$$

   The first part of the shape parameter $\alpha_0 > 0$ is a *constant* characterizing the proper dynamics of the system deterioration. The second part $\alpha\left(X_{E_j^+}\right)$, where $\alpha(0) = 0$, is a *continuous increasing non-negative function* of $X_{E_j^+}$, $0 \leq X_{E_j^+} < L$, representing the effect of past dependency on the future deterioration dynamics. Consequently, between two times s and t, $E_j \leq s < t \leq S_j$, the random deterioration increment $X_t - X_s$ is Gamma distributed with probability density function (pdf)

$$f_{\left(\alpha_0+\alpha\left(X_{E_j^+}\right)\right)\cdot(t-s),\beta}(x) = \frac{\beta^{\left(\alpha_0+\alpha\left(X_{E_j^+}\right)\right)\cdot(t-s)} x^{\left(\alpha_0+\alpha\left(X_{E_j^+}\right)\right)\cdot(t-s)-1} e^{-\beta x}}{\Gamma\left(\left(\alpha_0 + \alpha\left(X_{E_j^+}\right)\right)\cdot(t-s)\right)} \cdot 1_{\{x \geq 0\}}, \tag{2}$$

   and survival function

$$\bar{F}_{\left(\alpha_0+\alpha\left(X_{E_j^+}\right)\right)\cdot(t-s),\beta}(x) = \frac{\Gamma\left(\left(\alpha_0 + \alpha\left(X_{E_j^+}\right)\right)\cdot(t-s), \beta x\right)}{\Gamma\left(\left(\alpha_0 + \alpha\left(X_{E_j^+}\right)\right)\cdot(t-s)\right)}, \tag{3}$$

   where $1_{\{\cdot\}}$ denotes the indicator function which equals 1 if the argument is true and 0 otherwise, $\Gamma(\alpha) = \int_0^\infty z^{\alpha-1}e^{-z}dz$ and $\Gamma(\alpha, x) = \int_x^\infty z^{\alpha-1}e^{-z}dz$ are respectively the complete and upper incomplete gamma functions. The inspection is assumed perfect in the sense that it reveals the exact deterioration level of the system. Moreover, it takes negligible time, has no effect on the system deterioration, and incurs a constant unit cost $C_m > 0$.

2. At an inspection time $T_{j,k} = E_j + k \cdot \delta$, $k = 1, 2, \ldots$, a CBM decision rule based on both $X_{T_{j,k}}$ and $X_{E_j^+}$ is adopted.

(a) If $X_{T_{j,k}} \geq L$, a corrective replacement (CR) with constant unit cost C_c is carried out immediately on the failed system (i.e., $S_j = T_{j,k}$). It takes a constant duration λ_0 due to e.g., the material set-up, the system dismantling and reassembly. During the CR, the system is inactivated, so that the system deterioration level keeps unchanged (i.e., $X_{E^-_{j+1}} = X_{S_j}$, with $E_{j+1} = S_j + \lambda_0$). The system inactivity incurs a constant cost rate $C_i > 0$. Furthermore, before the CR starts, the system has been failed and unavailable until S_j. Such a system unavailability incurs a constant cost rate $C_u > C_i$ because it is unforeseen. After the CR, the system is AGAN such that $X_{E^+_{j+1}} = 0$.

(b) If $\zeta \leq X_{T_{j,k}} < L$ and $X_{E^+_j} \geq \eta$, a preventive replacement (PR) with constant unit cost $C_p < C_c$ is immediately carried out on the unrepairable badly deteriorated system (i.e., $S_j = T_{j,k}$). The PR also takes a constant duration λ_0 due to the same reason as above. Over $\left[S_j, E_{j+1}\right]$, with $E_{j+1} = S_j + \lambda_0$, the system is inactivated at the cost rate C_i and it deterioration level is unchanged (i.e., $X_{E^-_{j+1}} = X_{S_j}$). After the PR, the system is reset to be AGAN (i.e., $X_{E^+_{j+1}} = 0$).

(c) If $\zeta \leq X_{T_{j,k}} < L$ and $X_{E^+_j} < \eta$, a preventive partial repair (PPR) with constant unit cost $C_r \in \left(C_m, C_p\right)$ is immediately carried out on the repairable badly deteriorated system (i.e., $S_j = T_{j,k}$). The PPR requires a duration I_j depending on $X_{E^+_j}$ and X_{S_j} such that

$$I_j = \lambda_0 + \lambda\left(X_{E^+_j}, X_{S_j}\right), \tag{4}$$

where λ_0 is a constant duration as for replacements, $\lambda\left(X_{E^+_j}, X_{S_j}\right)$ is a continuous increasing function of $X_{E^+_j}$ and X_{S_j}, with $\lambda(0,0) = 0$. So, the higher the value of $X_{E^+_j}$ or X_{S_j}, the longer the required duration I_j for PPR. As in the case of replacements, the system is inactivated during the PPR at the cost rate C_i, and its deterioration level is still unchanged (i.e., $X_{E^-_{j+1}} = X_{S_j}$, with $E_{j+1} = S_j + I_j$). Just after the PPR, the system restarts with a deterioration level $X_{E^+_{j+1}} \in \left[X_{E^+_j}, X_{S_j}\right]$ sampled from a pdf truncated by X_{S_j} and $X_{E^+_j}$

$$X_{E^+_{j+1}} \sim g\left(y \mid X_{S_j}, X_{E^+_j}\right). \tag{5}$$

(d) If $X_{T_{j,k}} < \zeta$, no further intervention is needed at $T_{j,k}$, and hence nothing is changed. The decision is postponed to the next inspection at $T_{j,k+1} = T_{j,k} + \delta$.

The next cycle begins at E_{j+1} with initial deterioration level $X_{E^+_{j+1}}$. On which, the system deterioration evolves following a homogeneous Gamma process with shape parameter $\alpha_0 + \alpha\left(X_{E^+_{j+1}}\right)$ and scale parameter $\beta > 0$. If $X_{E^+_{j+1}} \geq X_{E^+_j}$ (it is true when the last maintenance action is a PPR), the system deteriorates with a higher average speed and a higher variance than before. Figure 1 illustrates the deterioration evolution of the CBMS over some first repair/replacement cycles.

For the considered system, the failure threshold L, and the maintenance costs C_m, C_r, C_p, C_c, C_i and C_u are input data. The scalars α_0, β and λ_0, and the functions $\alpha(\cdot)$, $\lambda(\cdot)$ and $g(\cdot \mid \cdot, \cdot)$ are parameters to be determined from deterioration and maintenance data. Although this problem has not been dealt with in this paper, we still believe that the following two-steps procedure could be applied. For some conjectured parametric forms of $\alpha(\cdot)$, $\lambda(\cdot)$ and $g(\cdot \mid \cdot, \cdot)$, classical methods (e.g., maximum likelihood method, method of moment, etc.) are used to estimate the model parameters (including α_0, β and λ_0) from deterioration and maintenance data. Next, we perform goodness-of-fit tests to find the best fit of the data. For this estimation-testing procedure, the deterioration and maintenance data are obviously prerequisite. This is why building such a data-set is recognized as a key perspective of the paper. Finally, the inspection period δ, and the deterioration thresholds ζ and η are decision variables to be jointly optimized. As argued in (Wagner, 1975, chapter 11), the long-run expected cost rate is a suitable objective function

$$C_\infty(\delta, \zeta, \eta) = \lim_{t\to\infty} \frac{C(t)}{t}, \tag{6}$$

where $C(t)$ denotes the cumulative maintenance cost incurred in the time interval $[0, t]$.

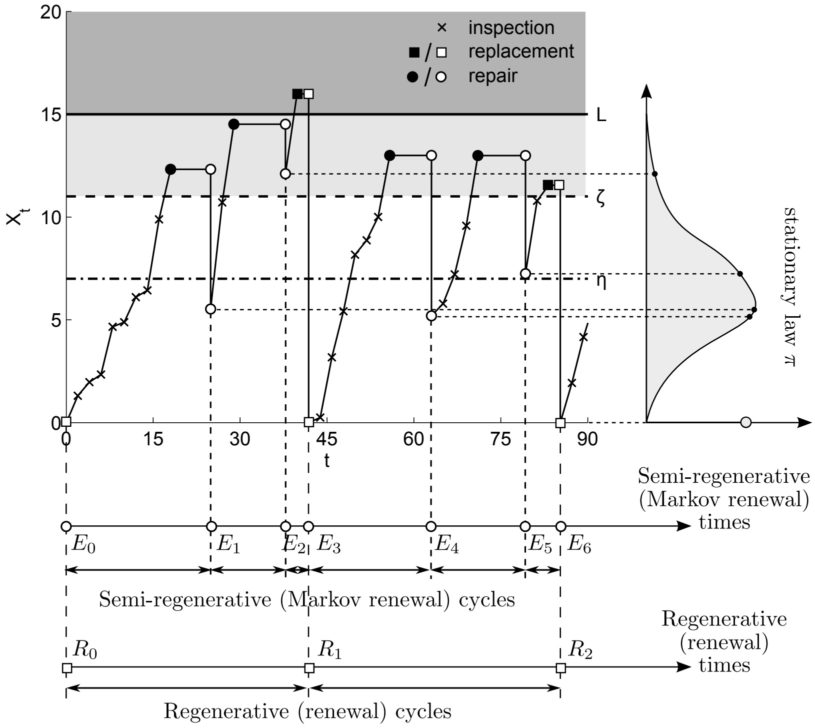


Figure 1: Schematic evolution of the maintained system state

To illustrate the practicalness of the proposed CBMS, let us introduce the gyroscope equipment represented in (Hu et al., 2018; Pei et al., 2018). Gyroscope is a core component in inertial navigation systems. Due to the wear of rotor spin axis and the friction of gimbal bearings, the gyroscopic drift increases over time, and hence degrades the gyroscope performance. Therefore, the drift can be seen as a deterioration index of the gyroscope. In the experiment provided by Hu et al. (2018); Pei et al. (2018), the gyroscope is periodically inspected for $\delta = 2.5$h each time. Whenever the gyroscopic drift revealed by an inspection exceeds a threshold $L = 0.37°/$h, the gyroscope is considered as failed and must be replaced. If the drift value is still less than $L = 0.37°/$h but greater than $\zeta = 0.30°/$h, the current in the torque coil of the gyroscope is adjusted to compensate the drift value. Such an adjustment is a partial repair action on the gyroscope. Compared to the decision rule implemented in the above CBMS, this one is a particular case with $\eta = L$. The evolution of the drift data of two maintained gyroscopes are plotted in Figure 2. Looking at the sequence of drift levels

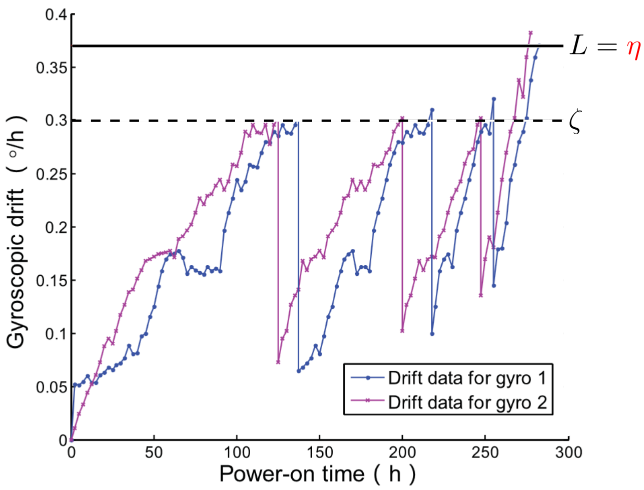


Figure 2: Drift data of two maintained gyroscopes adapted from (Hu et al., 2018; Pei et al., 2018)

after adjustments and the deterioration speed of the gyroscope, their characteristics is completely covered in our CBMS. Although the duration of maintenance actions has not been mentioned in this experiment, various existing works (see e.g., (Castanier et al., 2003; Liao et al., 2006)) confirm that the repair duration is very influenced by partial repairs.

3. Asymptotic deterioration behavior of the condition-based maintained system

Evaluating the maintenance cost rate in the long term (6) requires the study of the asymptotic deterioration behavior of the maintained system. In practice, the system does not wait till infinity to reach its asymptotic behavior, but rather at a finite moment from which the short-run maintenance cost rate converges to the long-run one with an acceptable error. Theoretically, the study of asymptotic behavior can be significantly simplified thanks to the semi-regenerative properties of the deterioration process of the maintained system (Bérenguer, 2008). In this section, after analyzing how these semi-regenerative properties simplifies the evaluation of (6), we derive the stationary law of the maintained system deterioration.

3.1. Semi-regenerative properties of the maintained system deterioration

The semi-regenerative properties of the deterioration process $\{X_t\}_{t\geq 0}$ of the maintained system allow studying the asymptotic deterioration behavior of the system on a limited horizon instead of an infinite horizon. After the end of a system repair/replacement at E_j, $j \in \mathbb{N}$, the future deterioration process $\left\{X_{t+E_j}\right\}_{t\geq 0}$ of the maintained system depends on its past $\{X_t\}_{0\leq t\leq E_j}$ only via $X_{E_j^+}$. Therefore, apart from its regenerative structure, $\{X_t\}_{t\geq 0}$ appears as a semi-regenerative process (Cinlar, 1975, page 343), with repair/replacement end times E_j^+, $j \in \mathbb{N}$, as semi-regenerative (Markov renewal) times (see Figure 1). Embedded in $\{X_t\}_{t\geq 0}$, there exits a Markov chain $\left\{Y_j\right\}_{j\in\mathbb{N}}$, $Y_j = X_{E_j^+}$, with stationary law π. As shown in (Grall et al., 2002), the study of asymptotic behavior of $\{X_t\}_{t\geq 0}$ can be restricted to a single semi-regenerative cycle (also known as Markov renewal cycle) defined by two successive repair/replacement times, and the long-run maintenance cost rate (6) can be expressed by

$$C_\infty(\delta,\zeta,\eta) = \lim_{t\to\infty}\frac{C(t)}{t} = \frac{E_\pi\left[C\left(\left[0^+, E_1^+\right]\right)\right]}{E_\pi\left[\Delta E_1\right]}, \tag{7}$$

where $E_\pi[\cdot]$ denotes the s-expectation with respect to the stationary π, and $\Delta E_1 = E_1$ denotes the length of the first Markov renewal cycle $\left[0^+, E_1^+\right]$. We note that 0^+ is not merely the initial working time of the system at which $X_{0^+} = 0$, but rather the beginning of a Markov renewal cycle at which $X_{0^+} = x$, where $0 \leq x < L$. Some elements of proof of (7) are provided in (Mercier and Pham, 2014). A short length of ΔE_1 simplifies the analysis of the system deterioration behavior, thence allows an analytical evaluation of $C_\infty(\delta,\zeta,\eta)$. However, the price paid to this simplicity is the derivation of the stationary law π of the embedded Markov chain $\left\{Y_j\right\}_{j\in\mathbb{N}}$, which is the main difficulty of this approach.

3.2. Stationary law of the embedded Markov chain $\left\{Y_j\right\}_{j\in\mathbb{N}}$

The embedded Markov chain $\left\{Y_j\right\}_{j\in\mathbb{N}}$ describes the system deterioration state at the end of a repair/replacement action. It starts from $Y_0 = 0$, takes the value in the continuous state space $[0, L)$, and comes back to 0 (i.e., a regeneration set) almost surely due to replacement actions (see Figure 1). Therefore, there exists a stationary measure π for $\left\{Y_j\right\}_{j\in\mathbb{N}}$ on $[0, L)$, which is the solution of the invariance equation (Asmussen and Glynn, 2007, page 97)

$$\pi(dy) = \int_{[0,L)} P(x, dy)\,\pi(dx), \tag{8}$$

where $P(x, dy)$ stands for the transition kernel of $\left\{Y_j\right\}_{j\in\mathbb{N}}$ from $X_{E_j^+} = x$ to $X_{E_{j+1}^+} = y$. In the following, we seek a closed-form expression of $P(x, dy)$, and thence propose a solution for (8).

3.2.1. Transition kernel $P(x, dy)$

A closed-form expression of $P(x, dy)$ can be obtained by exhaustively analyzing different possible scenarios of the maintained system under the (δ,ζ,η) policy from the beginning E_j^+ to the end E_{j+1}^+ of the j-th Markov renewal cycle. As shown in Appendix A, we can express $P(x, dy)$ as follows.

1. when $\zeta < \eta$,

$$P(x, dy) = \Big(\delta_0(dy) \cdot (\rho_1(x) + \rho_2(x)) + (p_1(y \mid x) + p_2(y \mid x)) \cdot 1_{\{y \neq 0, x \leq y < L\}} \cdot dy\Big) \cdot 1_{\{0 \leq x < \zeta\}} + \Big(\delta_0(dy) \cdot \rho_1(x) + p_3(y \mid x) \cdot 1_{\{y \neq 0, x \leq y < L\}} \cdot dy\Big) \cdot 1_{\{\zeta \leq x < \eta\}} + \delta_0(dy) \cdot 1_{\{\eta \leq x < L\}}, \quad (9)$$

2. when $\eta \leq \zeta$,

$$P(x, dy) = \Big(\delta_0(dy) \cdot (\rho_1(x) + \rho_2(x)) + (p_1(y \mid x) + p_2(y \mid x)) \cdot 1_{\{y \neq 0, x \leq y < L\}} \cdot dy\Big) \cdot 1_{\{0 \leq x < \eta\}} + (\rho_3(x) + \rho_4(x)) \cdot \delta_0(dy) \cdot 1_{\{\eta \leq x < \zeta\}} + \delta_0(dy) \cdot 1_{\{\zeta \leq x < L\}}, \quad (10)$$

where, given $X_{E_j^+} = x$,

- $\rho_1(x)$ denotes the conditional probability of a CR after one inspection period since E_j^+

$$\rho_1(x) = \bar{F}_{(\alpha_0 + \alpha(x))\delta, \beta}(L - x), \quad (11)$$

- $\rho_2(x)$ denotes the conditional probability of a CR after multiple inspection periods since E_j^+

$$\rho_2(x) = \int_x^\zeta \bar{F}_{(\alpha_0 + \alpha(x))\delta, \beta}(L - w) \sum_{k=1}^{\infty} f_{(\alpha_0 + \alpha(x))k\delta, \beta}(w - x)\, dw, \quad (12)$$

- $\rho_3(x)$ denotes the conditional probability of a PR after one inspection period since E_j^+

$$\rho_3(x) = \bar{F}_{(\alpha_0 + \alpha(x))\delta, \beta}(\zeta - x), \quad (13)$$

- $\rho_4(x)$ denotes the conditional probability of a PR after multiple inspection periods since E_j^+

$$\rho_4(x) = \int_x^\zeta \bar{F}_{(\alpha_0 + \alpha(x))\delta, \beta}(\zeta - w) \sum_{k=1}^{\infty} f_{(\alpha_0 + \alpha(x))k\delta, \beta}(w - x)\, dw, \quad (14)$$

- $p_1(y \mid x)$ stands for the conditional pdf of a PPR after one inspection period since E_j^+, with $x \in [0, \eta)$,

$$p_1(y \mid x) = \int_\zeta^L g(y \mid x, r)\, f_{(\alpha_0 + \alpha(x))\delta, \beta}(r - x)\, dr, \quad (15)$$

- $p_2(y \mid x)$ stands for the conditional pdf of a PPR after multiple inspection periods since E_j^+

$$p_2(y \mid x) = \int_x^\zeta \left(\int_\zeta^L g(y \mid x, z)\, f_{(\alpha_0 + \alpha(x))\delta, \beta}(z - w)\, dz \right) \sum_{k=1}^{\infty} f_{(\alpha_0 + \alpha(x))k\delta, \beta}(w - x)\, dw, \quad (16)$$

- $p_3(y \mid x)$ stands for the conditional pdf of a PPR after one inspection period since E_j^+, with $x \in [\eta, \zeta)$,

$$p_3(y \mid x) = \int_x^L g(y \mid x, r)\, f_{(\alpha_0 + \alpha(x))\delta, \beta}(r - x)\, dr, \quad (17)$$

in which $f_{(\alpha_0 + \alpha(x)) \cdot (\cdot), \beta}(\cdot)$ is given from (2) and $\bar{F}_{(\alpha_0 + \alpha(x))\delta, \beta}(\cdot)$ is derived from (3).

We note that the expression of $P(x, dy)$ consists of both the dirac part and continuous part. The magnitude of the Dirac measure located at 0 represents the probability of a replacement (either corrective or preventive) done during a Markov renewal cycle. Besides, $P(x, dy)$ does not impacted by the maintenance duration, in which the system deterioration keeps unchanged.

3.2.2. Solution for the stationary law $\pi(dy)$

As the expression of $P(x, dy)$, the stationary law $\pi(dy)$ is also a convex combination of Dirac mass function and a continuous pdf. Appendix B gives the mathematical expression of $\pi(dy)$ as follows.

1. When $\zeta < \eta$,

$$\pi(dy) = a \cdot \delta_0(dy) + (1-a) \cdot b_1(y) \cdot 1_{\{0<y<\zeta\}} dy + (1-a) \cdot b_2(y) \cdot 1_{\{\zeta \le y<\eta\}} dy + (1-a) \cdot b_3(y) \cdot 1_{\{\eta \le y<L\}} dy, \quad (18)$$

where

$$a = \frac{1}{1 + \int_0^\zeta B_1(y)\,dy + \int_\zeta^\eta B_2(y)\,dy + \int_\eta^L B_3(y)\,dy}, \quad (19)$$

and

$$b_1(y) = \frac{a}{1-a} \cdot B_1(y), \quad b_2(y) = \frac{a}{1-a} \cdot B_2(y), \quad \text{and} \quad b_3(y) = \frac{a}{1-a} \cdot B_3(y). \quad (20)$$

$B_1(y)$, $B_2(y)$ and $B_3(y)$ are computed by

- when $0 < y < \zeta$,

$$B_1(y) = p_1(y \mid 0) + p_2(y \mid 0) + \int_0^y B_1(x) \cdot (p_1(y \mid x) + p_2(y \mid x))\,dx, \quad (21)$$

- when $\zeta < y < \eta$,

$$B_2(y) = p_1(y \mid 0) + p_2(y \mid 0) + \int_0^\zeta B_1(x) \cdot (p_1(y \mid x) + p_2(y \mid x))\,dx + \int_\zeta^y B_2(x) \cdot p_3(y \mid x)\,dx, \quad (22)$$

- when $\eta < y < L$,

$$B_3(y) = p_1(y \mid 0) + p_2(y \mid 0) + \int_0^\zeta B_1(x) \cdot (p_1(y \mid x) + p_2(y \mid x))\,dx + \int_\zeta^\eta B_2(x) \cdot p_3(y \mid x)\,dx, \quad (23)$$

where $p_1(y \mid x)$, $p_2(y \mid x)$ and $p_3(y \mid x)$ are given from (15), (16) and (17).

2. When $\eta \le \zeta$,

$$\pi(dy) = c \cdot \delta_0(dy) + (1-c) \cdot d_1(y) \cdot 1_{\{0<y<\eta\}} dy + (1-c) \cdot d_2(y) \cdot 1_{\{\eta \le y<\zeta\}} dy + (1-c) \cdot d_3(y) \cdot 1_{\{\zeta \le y<L\}} dy, \quad (24)$$

where

$$c = \frac{1}{1 + \int_0^\eta D_1(y)\,dy + \int_\eta^\zeta D_3(y)\,dy + \int_\zeta^L D_3(y)\,dy}, \quad (25)$$

and

$$d_1(y) = \frac{c}{1-c} \cdot D_1(y), \quad d_2(y) = \frac{c}{1-c} \cdot D_2(y), \quad \text{and} \quad d_3(y) = \frac{c}{1-c} \cdot D_3(y). \quad (26)$$

$D_1(y)$, $D_2(y)$ and $D_3(y)$ are computed by

- when $0 < y < \eta$

$$D_1(y) = p_1(y \mid 0) + p_2(y \mid 0) + \int_0^y D_1(x) \cdot (p_1(y \mid x) + p_2(y \mid x))\,dx, \quad (27)$$

- when $\eta < y < \zeta$

$$D_2(y) = p_1(y \mid 0) + p_2(y \mid 0) + \int_0^\eta D_1(x) \cdot (p_1(y \mid x) + p_2(y \mid x))\,dx, \quad (28)$$

- when $\zeta < y < L$

$$D_3(y) = p_1(y \mid 0) + p_2(y \mid 0) + \int_0^{\eta} D_1(x) \cdot (p_1(y \mid x) + p_2(y \mid x))\, dx, \tag{29}$$

where $p_1(y \mid x)$ and $p_2(y \mid x)$ are given from (15) and (16).

Solving the non-homogeneous linear Volterra integral equations of the second kind (21), (22) and (27) allows to fully derive $\pi(dy)$. However, their analytical solutions are not easy to handle. To overcome this obstacle, the *Heun' s numerical method* (Kharab and Guenther, 2011, pages 334-335) is used to approximate the solution of (21), (22) and (27). Appendix B.2 gives the detail of this numerical method.

4. Cost-based optimization of the condition-based maintained system

This section aims at optimizing the considered CBMS using the long-run maintenance cost rate (6). To this end, we formulate a closed-form expression of $C_\infty(\delta, \zeta, \eta)$ following (7). Next, we apply derivative free algorithms for black-box optimization (e.g., generalized pattern search (Audet and Hare, 2017)) to $C_\infty(\delta, \zeta, \eta)$ to search the optimal decision parameters $\left(\delta_{opt}, \zeta_{opt}, \eta_{opt}\right)$.

4.1. Maintenance cost rate evaluation

From the decision structure implemented in the considered CBMS, we can express the cumulative cost incurred in the time interval $[0, t]$ as

$$C(t) = C_r \cdot N_r(t) + C_p \cdot N_p(t) + C_c \cdot N_c(t) + C_m \cdot N_m(t) + C_u \cdot U(t) + C_i \cdot I(t), \tag{30}$$

where $N_r(t)$, $N_p(t)$, $N_c(t)$ and $N_m(t)$ denote respectively the number of PPR, of PR, of CR, and of inspections in $[0, t]$, $U(t)$ and $I(t)$ stand for total duration of system unavailability and system inactivity in $[0, t]$. Using (7), we obtain the expression of $C_\infty(\delta, \zeta, \eta)$ as

$$\begin{aligned} C_\infty(\delta, \zeta, \eta) = \frac{E_\pi\left[C\left(\left[0^+, E_1^+\right]\right)\right]}{E_\pi[\Delta E_1]} &= C_r \cdot \frac{E_\pi\left[N_r\left(\left[0^+, E_1^+\right]\right)\right]}{E_\pi[\Delta E_1]} + C_p \cdot \frac{E_\pi\left[N_p\left(\left[0^+, E_1^+\right]\right)\right]}{E_\pi[\Delta E_1]} + \\ C_c \cdot \frac{E_\pi\left[N_c\left(\left[0^+, E_1^+\right]\right)\right]}{E_\pi[\Delta E_1]} &+ C_m \cdot \frac{E_\pi\left[N_m\left(\left[0^+, E_1^+\right]\right)\right]}{E_\pi[\Delta E_1]} + C_u \cdot \frac{E_\pi\left[U\left(\left[0^+, E_1^+\right]\right)\right]}{E_\pi[\Delta E_1]} + C_i \cdot \frac{E_\pi\left[I\left(\left[0^+, E_1^+\right]\right)\right]}{E_\pi[\Delta E_1]}. \end{aligned} \tag{31}$$

Since $\left[0^+, E_1^+\right] = [0^+, S_1] \cup \left[S_1, E_1^+\right]$, where $S_1 = \delta \cdot N_m\left(\left[0^+, E_1^+\right]\right)$, (31) can be rewritten as

$$\begin{aligned} C_\infty(\delta, \zeta, \eta) = &\frac{1}{\delta \cdot E_\pi\left[N_m\left(\left[0^+, E_1^+\right]\right)\right] + E_\pi\left[I\left(\left[0^+, E_1^+\right]\right)\right]} \cdot \Big(C_r \cdot E_\pi\left[N_r\left(\left[0^+, E_1^+\right]\right)\right] + C_p \cdot E_\pi\left[N_p\left(\left[0^+, E_1^+\right]\right)\right] \\ &+ C_c \cdot E_\pi\left[N_c\left(\left[0^+, E_1^+\right]\right)\right] + C_m \cdot E_\pi\left[N_m\left(\left[0^+, E_1^+\right]\right)\right] + C_u \cdot E_\pi\left[U\left(\left[0^+, E_1^+\right]\right)\right] + C_i \cdot E_\pi\left[I\left(\left[0^+, E_1^+\right]\right)\right]\Big). \end{aligned} \tag{32}$$

Hereinafter, we analyze all the possible maintenance scenarios on the first Markov renewal cycle $\left[0^+, E_1^+\right]$, and thence we derive mathematical expressions of $E_\pi\left[N_r\left(\left[0^+, E_1^+\right]\right)\right]$, $E_\pi\left[N_p\left(\left[0^+, E_1^+\right]\right)\right]$, $E_\pi\left[N_c\left(\left[0^+, E_1^+\right]\right)\right]$, $E_\pi\left[N_m\left(\left[0^+, E_1^+\right]\right)\right]$, $E_\pi\left[U\left(\left[0^+, E_1^+\right]\right)\right]$ and $E_\pi\left[I\left(\left[0^+, E_1^+\right]\right)\right]$. The exactness of the formulation is also justified by numerical experiments.

4.1.1. Possible maintenance scenarios on the first Markov renewal cycle

Let consider the Markov renewal cycle $\left[0^+, E_1^+\right]$ with $X_{0^+} = y$, the decision structure of (δ, ζ, η) policy leads to following possible maintenance scenarios.

1. When $\zeta < \eta$, then
   (a) the system maintenance starts after one inspection period δ since 0^+ (i.e., $S_1 = \delta$) by

- *scenario 1*: a PPR with duration $\lambda_0 + \lambda(X_{0^+}, X_\delta)$ if $\{0 \le X_{0^+} < \zeta \le X_\delta < L\}$ or $\{\zeta \le X_{0^+} < \eta, X_\delta < L\}$,
- *scenario 2*: a PR with duration λ_0 if $\{\eta \le X_{0^+} < X_\delta < L\}$,
- *scenario 3*: a CR with duration λ_0 if $\{0 \le X_{0^+} < L \le X_\delta\}$,

(b) the system maintenance starts after a multiple of inspection period $(k+1)\delta$, $k = 1, 2, \ldots$, since 0^+ (i.e., $S_1 = (k+1)\delta$) by

- *scenario 4*: a PPR with duration $\lambda_0 + \lambda(X_{0^+}, X_{(k+1)\delta})$ if $\{0 \le X_{0^+} \le X_{k\delta} < \zeta \le X_{(k+1)\delta} < L\}$,
- *scenario 5*: a CR with duration λ_0 if $\{0 \le X_{0^+} \le X_{k\delta} < \zeta < L \le X_{(k+1)\delta}\}$.

2. When $\eta \le \zeta$, then

(a) the system maintenance starts after one inspection period δ since 0^+ (i.e., $S_1 = \delta$) by

- *scenario 6*: a PPR with duration $\lambda_0 + \lambda(X_{0^+}, X_\delta)$ if $\{0 \le X_{0^+} < \eta \le \zeta \le X_\delta < L\}$,
- *scenario 7*: a PR with duration λ_0 if $\{\eta \le X_{0^+} < \zeta \le X_\delta < L\}$ or $\{\zeta \le X_{0^+} < X_\delta < L\}$,
- *scenario 8*: a CR with duration λ_0 if $\{0 \le X_{0^+} < L \le X_\delta\}$,

(b) the system maintenance starts after a multiple of inspection period $(k+1)\delta$, $k = 1, 2, \ldots$, since 0^+ (i.e., $S_1 = (k+1)\delta$) by

- *scenario 9*: a PPR with duration $\lambda_0 + \lambda(X_{0^+}, X_{(k+1)\delta})$ if $\{0 \le X_{0^+} \le X_{k\cdot\delta} < \zeta \le X_{(k+1)\delta} < L\}$
- *scenario 10*: a PR with duration λ_0 if $\{\eta \le X_{0^+} < X_{k\delta} < \zeta \le X_{(k+1)\delta} < L\}$,
- *scenario 11*: a CR with duration λ_0 if $\{0 \le X_{0^+} < \eta, X_{k\delta} < \zeta < L \le X_{(k+1)\delta}\}$ or $\{\eta \le X_{0^+} < X_{k\delta} < \zeta < L \le X_{(k+1)\delta}\}$.

The above scenarios are the basis to compute the required expectations.

4.1.2. Expected number of preventive partial repairs over the first Markov renewal cycle

As shown in Appendix C, we can express the expected value of $N_r\left(\left[0^+, E_1^+\right]\right)$ with respect to the stationary law π as

1. when $\zeta < \eta$,

$$E_\pi\left[N_r\left(0^+, E_1^+\right)\right] = a \cdot \left(\bar{F}_{\alpha_0\delta,\beta}(\zeta) - \bar{F}_{\alpha_0\delta,\beta}(L) + \int_0^\zeta \left(\bar{F}_{\alpha_0\delta,\beta}(\zeta - w) - \bar{F}_{\alpha_0\delta,\beta}(L - w)\right) \sum_{k=1}^{\infty} f_{\alpha_0 k\delta,\beta}(w)\, dw\right) + (1-a) \cdot \int_0^\zeta \left(\bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(\zeta - y) - \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L - y) + \int_y^\zeta \left(\bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(\zeta - w) - \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L - w)\right) \times \sum_{k=1}^{\infty} f_{(\alpha_0+\alpha(y))k\delta,\beta}(w - y)\, dw\right) b_1(y)\, dy + (1-a) \cdot \int_\zeta^\eta F_{(\alpha_0+\alpha(y))\delta,\beta}(L - y)\, b_2(y)\, dy, \quad (33)$$

where a, $b_1(y)$ and $b_2(y)$ are given from (19) and (20);

2. when $\eta \le \zeta$,

$$E_\pi\left[N_r\left(0^+, E_1^+\right)\right] = c \cdot \left(\bar{F}_{\alpha_0\delta,\beta}(\zeta) - \bar{F}_{\alpha_0\delta,\beta}(L) + \int_0^\zeta \left(\bar{F}_{\alpha_0\delta,\beta}(\zeta - w) - \bar{F}_{\alpha_0\delta,\beta}(L - w)\right) \sum_{k=1}^{\infty} f_{\alpha_0 k\delta,\beta}(w)\, dw\right) + (1-c) \cdot \int_0^\eta \left(\bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(\zeta - y) - \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L - y) + \int_y^\zeta \left(\bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(\zeta - w) - \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L - w)\right) \sum_{k=1}^{\infty} f_{(\alpha_0+\alpha(y))k\delta,\beta}(w - y)\, dw\right) d_1(y)\, dy, \quad (34)$$

where c and $d_1(y)$ are given from (25) and (26).

4.1.3. Expected number of preventive replacements over the first Markov renewal cycle

As shown in Appendix D, the expected value of $N_p\left(\left[0^+, E_1^+\right]\right)$ with respect to the stationary law π is given by

1. when $\zeta < \eta$,

$$E_\pi\left[N_p\left(0^+, E_1^+\right)\right] = (1-a)\cdot\int_\eta^L F_{(\alpha_0+\alpha(y))\delta,\beta}(L-y)\, b_3(y)\, dy, \tag{35}$$

where a and $b_3(y)$ are given from (19) and (20);

2. when $\eta \le \zeta$,

$$E_\pi\left[N_p\left(0^+, E_1^+\right)\right] = (1-c)\cdot\Bigg(\int_\eta^\zeta \Bigg(\bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(\zeta-y) - \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L-y) + \int_y^\zeta \left(\bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(\zeta-w) - \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L-w)\right)\sum_{k=1}^{\infty} f_{(\alpha_0+\alpha(y))k\delta,\beta}(w-y)\, dw\Bigg) d_2(y)\, dy + \int_\zeta^L F_{(\alpha_0+\alpha(y))\delta,\beta}(L-y)\, d_3(y)\, dy\Bigg), \tag{36}$$

where c, $d_2(y)$ and $d_3(y)$ are given from (25) and (26).

4.1.4. *Expected number of corrective replacements over the first Markov renewal cycle*

Appendix E gives the expected value of $N_c\left(\left[0^+, E_1^+\right]\right)$ with respect to the stationary law π as

1. when $\zeta < \eta$,

$$E_\pi\left[N_c\left(0^+, E_1^+\right)\right] = a\cdot\left(\bar{F}_{\alpha_0\delta,\beta}(L) + \int_0^\zeta \bar{F}_{\alpha_0\delta,\beta}(L-w)\sum_{k=1}^{\infty} f_{\alpha_0 k\delta,\beta}(w)\, dw\right) + (1-a)\times \Bigg(\int_0^\zeta\left(\bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L-y) + \int_y^\zeta \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L-w)\sum_{k=1}^{\infty} f_{(\alpha_0+\alpha(y))k\delta,\beta}(w-y)\, dw\right) b_1(y)\, dy + \int_\zeta^\eta \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L-y)\, b_2(y)\, dy + \int_\eta^L \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L-y)\, b_3(y)\, dy\Bigg), \tag{37}$$

where a, $b_1(y)$, $b_2(y)$ and $b_3(y)$ are given from (19) and (20);

2. when $\eta \le \zeta$,

$$E_\pi\left[N_c\left(0^+, E_1^+\right)\right] = c\cdot\left(\bar{F}_{\alpha_0\delta,\beta}(L) + \int_0^\zeta \bar{F}_{\alpha_0\delta,\beta}(L-w)\sum_{k=1}^{\infty} f_{\alpha_0 k\delta,\beta}(w)\, dw\right) + (1-c)\times \Bigg(\int_0^\eta\left(\bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L-y) + \int_y^\zeta \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L-w)\sum_{k=1}^{\infty} f_{(\alpha_0+\alpha(y))k\delta,\beta}(w-y)\, dw\right) d_1(y)\, dy + \int_\eta^\zeta\left(\bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L-y) + \int_y^\zeta \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L-w)\sum_{k=1}^{\infty} f_{(\alpha_0+\alpha(y))k\delta,\beta}(w-y)\, dw\right) d_2(y)\, dy + \int_\zeta^L \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(L-y)\, d_3(y)\, dy\Bigg), \tag{38}$$

where a, $d_1(y)$, $d_2(y)$ and $d_3(y)$ are given from (25) and (26).

4.1.5. *Expected number of inspections over the first Markov renewal cycle*

Based on the expectations of the scenarios derived in Appendix C, Appendix D and Appendix E, the expected value of $N_m\left(\left[0^+, E_1^+\right]\right)$ with respect to the stationary law π is obtained by

1. when $\zeta < \eta$,

$$E_\pi\left[N_m\left(\left[0^+, E_1^+\right]\right)\right] = a \cdot \bar{F}_{\alpha_0\delta,\beta}(\zeta) + (1-a)\cdot\left(\int_0^\zeta \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(\zeta - y)\, b_1(y)\, dy + \int_\zeta^\eta b_2(y)\, dy + \int_\eta^L b_3(y)\, dy\right)$$
$$+ a \cdot \int_0^\zeta \bar{F}_{\alpha_0\delta,\beta}(\zeta - w)\left(\sum_{k=1}^\infty (k+1)\, f_{\alpha_0 k\delta,\beta}(w)\right) dw + (1-a)\times$$
$$\int_0^\zeta \left(\int_y^\zeta \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(\zeta - w)\left(\sum_{k=1}^\infty (k+1)\, f_{(\alpha_0+\alpha(y))k\delta,\beta}(w-y)\right) dw\right) b_1(y)\, dy, \quad (39)$$

where a, $b_1(y)$, $b_2(y)$ and $b_3(y)$ are given from (19) and (20);

2. when $\eta \le \zeta$,

$$E_\pi\left[N_m\left(\left[0^+, E_1^+\right]\right)\right] = c \cdot \bar{F}_{\alpha_0\delta,\beta}(\zeta) + (1-c)\cdot\left(\int_0^\eta \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(\zeta - y)\, d_1(y)\, dy +\right.$$
$$\left.\int_\eta^\zeta \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(\zeta - y)\, d_2(y)\, dy + \int_\zeta^L d_3(y)\, dy\right) + c \cdot \int_0^\zeta \bar{F}_{\alpha_0\delta,\beta}(\zeta - w)\left(\sum_{k=1}^\infty (k+1)\, f_{\alpha_0 k\delta,\beta}(w)\right) dw$$
$$+ (1-c)\cdot \int_0^\eta \left(\int_y^\zeta \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(\zeta - w)\left(\sum_{k=1}^\infty (k+1)\, f_{(\alpha_0+\alpha(y))k\delta,\beta}(w-y)\right) dw\right) d_1(y)\, dy$$
$$+ (1-c)\cdot \int_\eta^\zeta \left(\int_y^\zeta \bar{F}_{(\alpha_0+\alpha(y))\delta,\beta}(\zeta - w)\left(\sum_{k=1}^\infty (k+1)\, f_{(\alpha_0+\alpha(y))k\delta,\beta}(w-y)\right) dw\right) d_2(y)\, dy, \quad (40)$$

where a, $d_1(y)$, $d_2(y)$ and $d_3(y)$ are given from (25) and (26).

4.1.6. Expected duration of the system unavailability over the first Markov renewal cycle

Appendix F shows the expected value of $U\left(\left[0^+, E_1^+\right]\right)$ with respect to the stationary law π as

1. when $\zeta < \eta$,

$$E_\pi\left[U\left(\left[0^+, E_1^+\right]\right)\right] = \int_0^\delta \left(a \cdot \left(\bar{F}_{\alpha_0 t,\beta}(L) + \int_0^\zeta \bar{F}_{\alpha_0 t,\beta}(L-w) \sum_{k=1}^\infty f_{\alpha_0 k\delta,\beta}(w)\, dw\right)\right.$$
$$+ (1-a)\cdot\left(\int_0^\zeta \left(\bar{F}_{(\alpha_0+\alpha(y))t,\beta}(L-y) + \int_y^\zeta \bar{F}_{(\alpha_0+\alpha(y))t,\beta}(L-w) \sum_{k=1}^\infty f_{(\alpha_0+\alpha(y))k\delta,\beta}(w-y)\, dw\right) b_1(y)\, dy\right.$$
$$\left.\left. + \int_\zeta^\eta \bar{F}_{(\alpha_0+\alpha(y))t,\beta}(L-y)\, b_2(y)\, dy + \int_\eta^L \bar{F}_{(\alpha_0+\alpha(y))t,\beta}(L-y)\, b_3(y)\, dy\right)\right) dt, \quad (41)$$

where a, $b_1(y)$, $b_2(y)$ and $b_3(y)$ are given from (19) and (20);

2. when $\eta \le \zeta$,

$$E_\pi\left[U\left(\left[0^+, E_1^+\right]\right)\right] = \int_0^\delta \left(c \cdot \left(\bar{F}_{\alpha_0 t,\beta}(L) + \int_0^\zeta \bar{F}_{\alpha_0 t,\beta}(L-w) \sum_{k=1}^\infty f_{\alpha_0 k\delta,\beta}(w)\, dw\right)\right.$$
$$+ (1-c)\cdot\left(\int_0^\eta \left(\bar{F}_{(\alpha_0+\alpha(y))t,\beta}(L-y) + \int_y^\zeta \bar{F}_{(\alpha_0+\alpha(y))t,\beta}(L-w) \sum_{k=1}^\infty f_{(\alpha_0+\alpha(y))k\delta,\beta}(w-y)\, dw\right) d_1(y)\, dy\right.$$
$$+ \int_\eta^\zeta \left(\bar{F}_{(\alpha_0+\alpha(y))t,\beta}(L-y) + \int_y^\zeta \bar{F}_{(\alpha_0+\alpha(y))t,\beta}(L-w) \sum_{k=1}^\infty f_{(\alpha_0+\alpha(y))k\delta,\beta}(w-y)\, dw\right) d_2(y)\, dy$$
$$\left.\left. + \int_\zeta^L \bar{F}_{(\alpha_0+\alpha(y))t,\beta}(L-y)\, d_3(y)\, dy\right)\right) dt, \quad (42)$$

where a, $d_1(y)$, $d_2(y)$ and $d_3(y)$ are given from (25) and (26).

4.1.7. Expected duration of the system inactivity over the first Markov renewal cycle

As proved in Appendix G, the expected duration of the system inactivity $E_\pi\left[I\left(\left[0^+, E_1^+\right]\right)\right]$ can be computed by

$$E_\pi\left[I\left(\left[0^+, E_1^+\right]\right)\right] = \lambda_0 + E_\pi\left[I_1\left(\left[0^+, E_1^+\right]\right)\right], \tag{43}$$

in which

1. when $\zeta < \eta$,

$$E_\pi\left[I_1\left(\left[0^+, E_1^+\right]\right)\right] = a \cdot \left(\int_\zeta^L \lambda(0, w)\, f_{\alpha_0\delta,\beta}(w)\, dw + \int_0^\zeta \left(\int_\zeta^L \lambda(0, v)\, f_{\alpha_0\delta,\beta}(v - w)\, dv\right) \sum_{k=1}^\infty f_{\alpha_0 k\delta,\beta}(w)\, dw\right) + (1 - a) \times \int_0^\zeta \left(\int_\zeta^L \lambda(y, w)\, f_{(\alpha_0+\alpha(y))\delta,\beta}(w - y)\, dw + \int_y^\zeta \left(\int_\zeta^L \lambda(y, v)\, f_{(\alpha_0+\alpha(y))\delta,\beta}(v - w)\, dv\right) \sum_{k=1}^\infty f_{(\alpha_0+\alpha(y))k\delta,\beta}(w - y)\, dw\right) \times b_1(y)\, dy + (1 - a) \cdot \int_\zeta^\eta \left(\int_y^L \lambda(y, w)\, f_{(\alpha_0+\alpha(y))\delta,\beta}(w - y)\, dw\right) b_2(y)\, dy, \tag{44}$$

   where a, $b_1(y)$ and $b_2(y)$ are given from (19) and (20);

2. when $\eta \le \zeta$,

$$E_\pi\left[I_1\left(\left[0^+, E_1^+\right]\right)\right] = c \cdot \left(\int_\zeta^L \lambda(0, w)\, f_{\alpha_0\delta,\beta}(w)\, dw + \int_0^\zeta \left(\int_\zeta^L \lambda(0, v)\, f_{\alpha_0\delta,\beta}(v - w)\, dv\right) \sum_{k=1}^\infty f_{\alpha_0 k\delta,\beta}(w)\, dw\right) + (1 - c) \times \int_0^\eta \left(\int_\zeta^L \lambda(y, w)\, f_{(\alpha_0+\alpha(y))\delta,\beta}(w - y)\, dw + \int_y^\zeta \left(\int_\zeta^L \lambda(y, v)\, f_{(\alpha_0+\alpha(y))\delta,\beta}(v - w)\, dv\right) \sum_{k=1}^\infty f_{(\alpha_0+\alpha(y))k\delta,\beta}(w - y)\, dw\right) \times d_1(y)\, dy, \tag{45}$$

   where c and $d_1(y)$ are given from (25) and (26).

4.1.8. Maintenance cost model validation

To validate the above mathematical formulation, we effectuate numerical comparisons between the results given by the numerical computation and the Monte Carlo simulation of $E_\pi\left[N_r\left(\left[0^+, E_1^+\right]\right)\right]$, $E_\pi\left[N_p\left(\left[0^+, E_1^+\right]\right)\right]$, $E_\pi\left[N_c\left(\left[0^+, E_1^+\right]\right)\right]$, $E_\pi\left[N_m\left(0^+, E_1^+\right)\right]$, $E_\pi\left[U\left(0^+, E_1^+\right)\right]$, $E_\pi\left[I\left(0^+, E_1^+\right)\right]$, as well as of $C_\infty(\delta, \zeta, \eta)$. For the numerical computation, we use the well-known trapezoidal rule to approximate integrals in the considered expectations. We also propose a simple way to derive the simulated results by Monte Carlo approach in Appendix H. In the following, an illustration is given on the basis of the maintained system defined by the set of parameters $L = 15$, $\alpha_0 = 1$, $\alpha\left(X_{E_j^+}\right) = 0.1 \cdot X_{E_j^+}$, $\beta = 1$, $\lambda_0 = 1$, $\lambda\left(X_{E_j^+}, X_{S_j}\right) = 0.1 \cdot X_{E_j^+} + 0.2 \cdot X_{S_j}$, and a continuous uniform pdf for $g(y \mid x, r)$, under two configurations of the (δ, ζ, η) policy

- *configuration 1* ($\zeta < \eta$): $\delta = 4$, $\zeta = 7$, $\eta = 11$,
- *configuration 2* ($\eta \le \zeta$): $\delta = 4$, $\eta = 7$, $\zeta = 11$.

The set of maintenance costs is chosen as $C_m = 5$, $C_r = 10$, $C_p = 100$, $C_c = 150$, $C_i = 5$ and $C_u = 25$. The duration T for Monte Carlo simulation (see Appendix H) is 10^8 time units. The results are shown as in Table 1. The almost identical results given by both the approaches justify the correctness of the developed mathematical cost model.

4.2. Optimum existence and searching

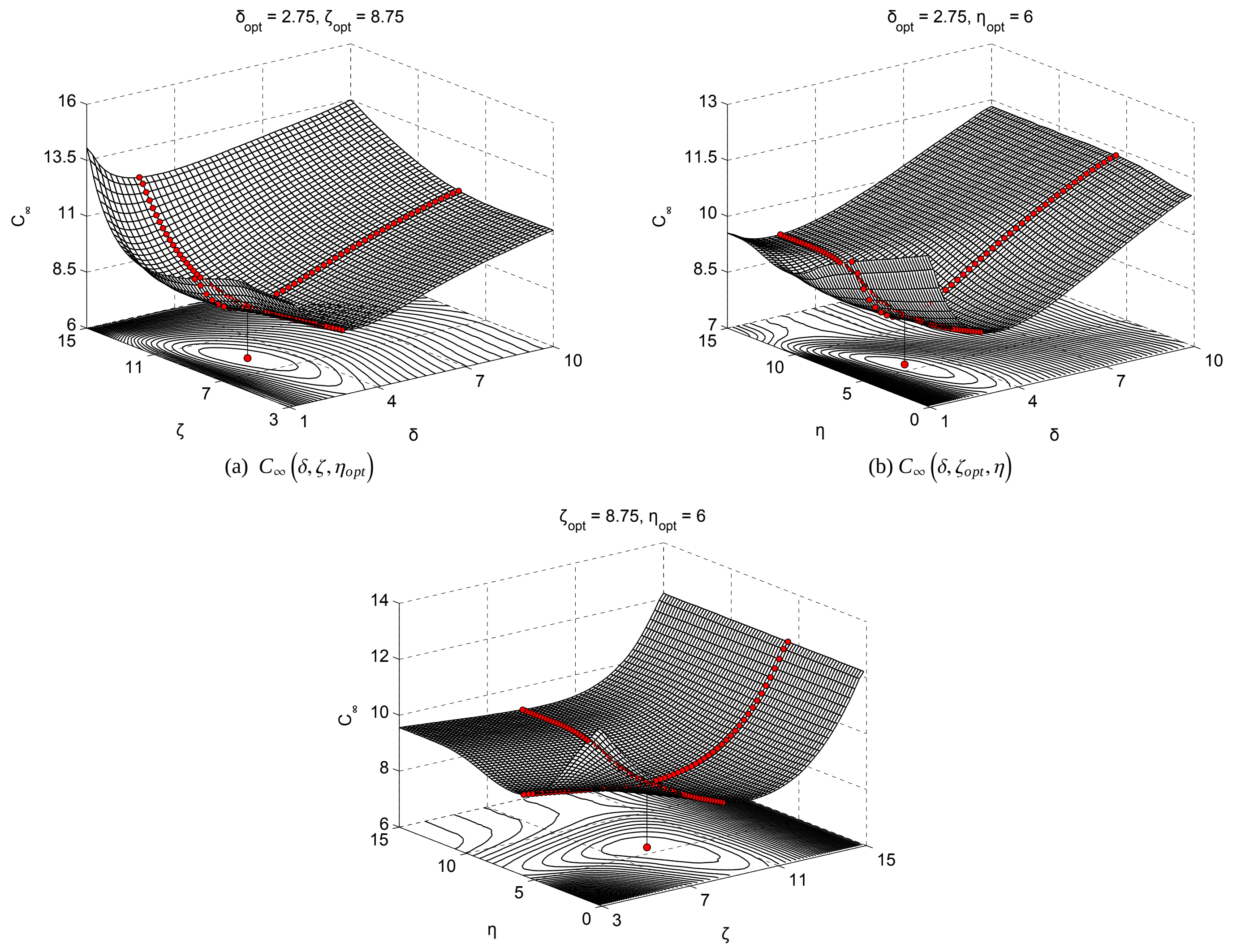
Optimizing the (δ, ζ, η) policy is to seek the triplet of decision parameters $\left(\delta_{opt}, \zeta_{opt}, \eta_{opt}\right)$ that minimizes $C_\infty(\delta, \zeta, \eta)$

$$C_\infty\left(\delta_{opt}, \zeta_{opt}, \eta_{opt}\right) = \min_{(\delta,\zeta,\eta)} \left\{C_\infty(\delta, \zeta, \eta), \delta > 0, 0 \le \zeta \le L, 0 \le \eta \le L\right\}. \tag{46}$$

Config.	Approach	$E_\pi[N_r]$	$E_\pi\left[N_p\right]$	$E_\pi[N_c]$	$E_\pi[N_m]$	$E_\pi[U]$	$E_\pi[I]$	$C_\infty(\delta,\zeta,\eta)$
1	Num. Comp.	0.7170	$6.7549 \cdot 10^{-4}$	0.2823	1.4897	0.4504	2.7185	9.2502
	M.C. Sim.	0.7175	$6.6489 \cdot 10^{-4}$	0.2819	1.4898	0.4486	2.7199	9.2312
2	Num. Comp.	0.5451	0.0635	0.3914	2.3377	0.5876	2.4599	9.4366
	M.C. Sim.	0.5458	0.0635	0.3907	2.3357	0.5851	2.4617	9.4227

Table 1: Results for validation the maintenance cost model

Analytical proof of optimum existence for the (δ,ζ,η) policy is unfeasible due to the complexity of the mathematical expression of $C_\infty(\delta,\zeta,\eta)$. To remedy this obstacle, we propose observing the shapes of $C_\infty(\delta,\zeta,\eta)$ when δ, ζ and η vary in a wide rank. Repeating observations for various configurations of system characteristics (i.e., for different α_0, $\alpha\left(X_{E_j^+}\right)$, β, L, $g_{X_{R_j^+}}$, λ_0 and $\lambda\left(X_{E_j^+}, S_j\right)$) and maintenance costs (i.e., C_m, C_r, C_p, C_c, C_i and C_u) allows us to confirm the existence of $\left(\delta_{opt},\zeta_{opt},\eta_{opt}\right)$. Even if this approach cannot cover all possible configurations, it is still an acceptable solution when analytical approach is impossible. A general conclusion drawn from these observations is that each of decision parameters δ, ζ and η has its own effect on the maintenance cost rate, and they have to be jointly optimized to achieve the best performance for the (δ,ζ,η) policy. As an illustration, we sketch in Figure (3) the shapes of $C_\infty(\delta,\zeta,\eta)$ for the same system considered in the example of Subsection 4.1. The convex forms of $C_\infty(\delta,\zeta,\eta)$ affirm the existence of $\left(\delta_{opt},\zeta_{opt},\eta_{opt}\right)$.



(a) $C_\infty\left(\delta,\zeta,\eta_{opt}\right)$

(b) $C_\infty\left(\delta,\zeta_{opt},\eta\right)$

(c) $C_\infty\left(\delta_{opt},\zeta,\eta\right)$

Figure 3: Shapes of $C_\infty(\delta,\zeta,\eta)$

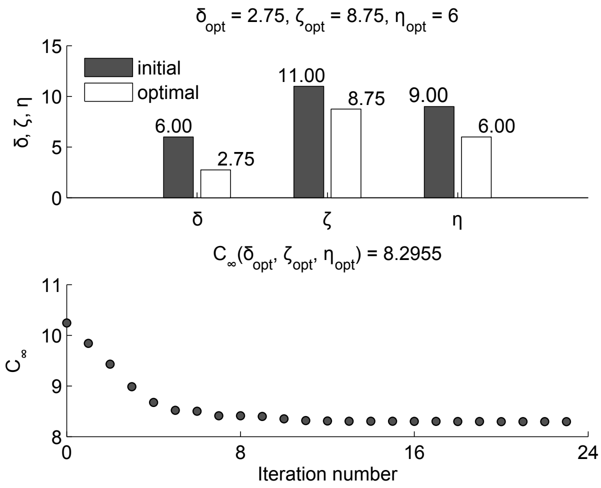


Figure 4: Optimization with Matlab's patternsearch solver

The triplet $\left(\delta_{opt},\zeta_{opt},\eta_{opt}\right)$ and the associated cost rate $C_\infty\left(\delta_{opt},\zeta_{opt},\eta_{opt}\right)$ can be found by the generalized pattern search algorithm (Audet and Hare, 2017). Indeed, continue with the above example, we obtain $\delta_{opt}=2.75$, $\zeta_{opt}=8.75$, $\eta_{opt}=6$ and $C_\infty\left(\delta_{opt},\zeta_{opt},\eta_{opt}\right)=8.2955$ when applying the *patternsearch* solver of Matlab's Global Optimization Toolbox to (32) (see Figure (4)). The generalized pattern search algorithm allows to find quickly the optimal configuration of the (δ,ζ,η) policy.

5. Numerical assessment of the condition-based maintained system

This section aims at using numerical experiments to confirm the effectiveness of the proposed CBMS, and to understand more deeply the impacts of the past-dependent PPR on the economic performance of the maintained system. To this end, we perform comparative studies of optimal long-run cost rate between the considered system (δ,ζ,η) and its two extreme cases (i.e., system with pure PR $(\delta,\zeta,\eta=0)$ (see e.g., (Huynh et al., 2011)), and system with pure PPR $(\delta,\zeta,\eta=L)$ (see e.g., (Meier-Hirmer et al., 2009))). Numerous experiments have been done for divers configurations of system characteristics and maintenance costs. However, illustrations shown in this section are just given from the maintained system characterized by

- a Gamma deterioration process with linear shape parameter $\text{HGP}\left(\alpha_0+\alpha\left(X_{E_j^+}\right),\beta\right)=\text{HGP}\left(\alpha_0+\alpha_1\cdot X_{E_j^+},\beta\right)$,
- a linear duration for PPR $\lambda_0+\lambda\left(X_{E_j^+},X_{S_j}\right)=\lambda_0+\lambda_1\cdot X_{E_j^+}+\lambda_2\cdot X_{S_j}$,
- a continuous uniform pdf for $g\left(y\mid x,r\right)$,

where $\alpha_0=1$, $\beta=1$, $\lambda_0=1$, $\lambda_2=0.25$, and $L=15$. The applied maintenance costs are $C_m=5$, $C_r=10$, $C_c=150$, $C_i=10$ and $C_u=25$. The choice of these values are completely arbitrary. The values of other parameters (i.e., α_1, λ_1) and maintenance cost (i.e., C_p) will be stated latter depending on specific sensitivity studies.

5.1. *Economic performance of the condition-based maintained system*

The distinction in economic performance of the three considered maintained systems comes from the difference between the PR cost C_p and the PPR cost C_r. Therefore, to see how good the system (δ,ζ,η) is, we vary the cost ratio $\frac{C_p}{C_r}$ in a wide range (C_r has been already fixed), and compare its optimal long-run cost rates with the other systems. The result shown in Figure 5 is obtained when fixing $\alpha_1=0.1$, $\lambda_1=0.25$ and varying $\frac{C_p}{C_r}$ from 3 to 15 with step 0.5.

The figure shows clearly that the system (δ,ζ,η) always saves more maintenance cost and returns to either the system $(\delta,\zeta,\eta=0)$ or the system $(\delta,\zeta,\eta=L)$ in worse cases. Indeed, it is equivalent to the former when C_p is relatively small, and to the latter when C_p is very high. The system (δ,ζ,η) reaches its best profit at a medium value of the ratio $\frac{C_p}{C_r}$. From the economic aspect, this result confirms that there is no risk in using the proposed CBMS compared to more classical ones.

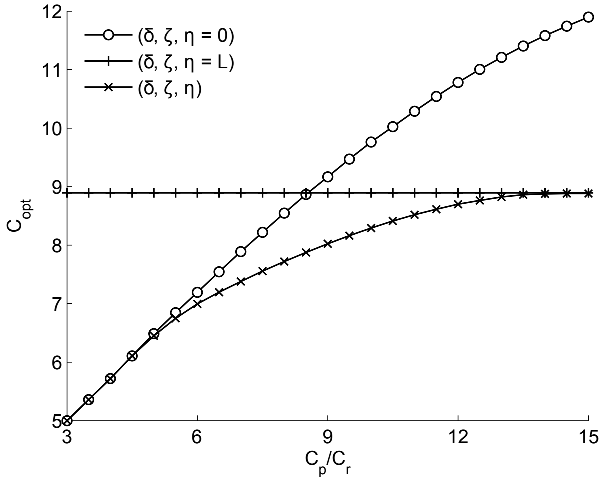


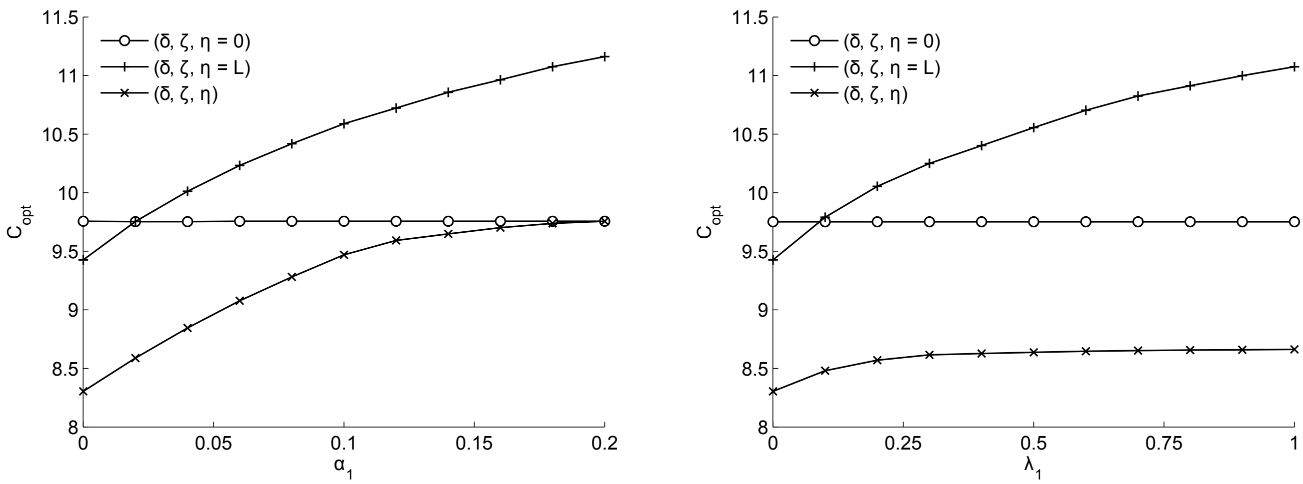
Figure 5: Sensitivity to $\frac{C_p}{C_r}$

5.2. Impacts of the past-dependent preventive partial repairs

To better understand the impacts of past-dependent PPR on the economic performance of the three maintained systems, we study separately how α_1 and λ_1 affect the evolution their optimal long-run maintenance cost rate. Consequently, two following configurations have been considered

1. α_1 is varied from 0 to 0.2 with step 0.025, and λ_1 is fixed at 0,
2. α_1 is fixed at 0, and λ_1 is varied from 0 to 0.9 with step 0.1.

We note that only one impact of the past-dependent PPR (either on the system deterioration dynamics via α_1 or on the repair duration via λ_1) is taken into account in each above configuration. The increasing α_1 and λ_1 imply the more and more important impacts of the past-dependent PPR. The PR cost is assumed fixed at $C_p = 100$. The results for the two above configurations are reported in Figures 6a and 6b.



(a) Sensitivity to α_1 (b) Sensitivity to λ_1

Figure 6: Impacts of the past-dependent preventive partial repairs

Obviously, the optimal long-run cost rate of the system $(\delta, \zeta, \eta = 0)$ is constant for both the considered configurations, because its maintenance decision structure is independent of PPR. Whereas, using the past-dependent partial repairs as a preventive action, the system (δ, ζ, η) and the system $(\delta, \zeta, \eta = L)$ incur higher maintenance cost due to the increasing of α_1 and λ_1. However, looking at the growth of their optimal long-run cost rate, the former resists the negative effects of the PPR much better than the latter. In other words, these negative effects can be significantly reduced if we combine properly the PPR and the PR into a CBM decision rule.

6. Conclusions and perspectives

The main focus of this paper is to model and evaluate the impacts of the past-dependent PPR on the economic performance of a condition-based maintained deteriorating system. A complete procedure, including the system deterioration modeling, the maintenance effects modeling, the elaboration of CBM decision rule, the formulation and optimization of mathematical cost model, and the maintenance model assessment, has been performed. Numerous numerical experiments confirm that the negative effects of the past-dependent PPR on the economic performance of the maintained system are unavoidable, but can be significantly reduced by coordinating the PPR and the PR into a CBM decision rule. The deterioration-based maintenance policy developed in this paper could be a good candidate. In fact, there is no risk when using the proposed CBMS, because it achieves at least the same cost-savings as the systems with pure PPR or with pure PR.

Given encouraging theoretical results, our next work is to valid the proposed CBM model with real-world data. A phase of data analysis, parameters estimation, and of models selection for system deterioration process and past-dependent repairs will be implemented before going further with the CBM decision rule. Currently, only the actual system deterioration state is used to make a maintenance decision. Meanwhile, the growing development of prognostics and health management techniques allows us to further access the information about the future system deterioration state (Lee et al., 2014). So, one of our perspectives is to study how this kind of information can be integrated in the CBM model to enable maintenance cost reduction. Using a prognostic-based inspection scheme instead of a periodic one could be an improved idea for the considered CBMS. Another perspective is to develop joint models of CBM and spare parts ordering for the considered system. This will remedy a strong assumption in the present CBMS that spare parts are always available for replacement actions.

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